Stochastic mechanical systems with unilateral state constraints: control prospects

Svetlana V. Anulova

Institute for Control Sciences, Moscow, Russia

1. There are two important intensively investigated fields in the theory of mechanical systems: systems with unilateral state constraints [1] and systems with random perturbations [2]. We begin a pioneering work in the overlap of these fields.

2. Consider a system, describing a particle moving in the axis with specular reflection at $\{0\}$: $t \geq 0$,

$$dX_t = Y_t\,dt + d\psi_t, \qquad X_t \geq 0, \qquad X_0 = x \geq 0,$$
$$dY_t = b(X_t, Y_t)\,dt + dW_t + d\varphi_t, \qquad Y_{0-} = y; \tag{1}$$
$$d\psi_t \geq 0,$$
$$\Delta\psi_t = 0, \qquad \psi_0 = 0; \tag{2}$$
$$d\phi_t = d\phi_t I_{\{0\}}(X_t) \geq 0,$$
$$\Delta\phi_t = 2|Y_{t-}| I_{(-\infty,0]}(Y_{t-}), \qquad \phi_{0-} = 0. \tag{3}$$

Here W is a random perturbation — the Wiener process — and ϕ is a random nondecreasing process, describing the reflection impacts on the velocity. The process ψ has no physical interpretation. Probably it equals zero, but it is not proven. At present we know only that $d\psi_t = d\psi_t I_{\{0,0\}}(X_t, Y_t)$.

The problem (1), (2) does not fit into the pattern of reflected diffusions in sense of Skorokhod.

Our ultimate aim is to transfer the results of [3] to the object (1)-(3). [3] consideres a control problem for a process in a convex domain, reflected in sense of Skorokhod. The payoff functional has infinite horizon and time discount. The problems of existence and optimization for the process are solved (Theorem 2.2 [3]) by a penalization method, which gives great opportunities for numerical calculations. The specular reflection operator is far and away more difficult to work with than the Skorokhod one. At present we do not know how to apply the penalization, and our achievement is the proof of existence of a solution of (1)-(3) by another method.

Theorem 1. *Assume that b is continuous and of no more than linear growth. Then for any initial conditions the solution exists (a weak one, in the sense of distributions).*

The proof uses the approach for SDE with reflecting boundaries in sense of Skorokhod: ϕ is approximated with a special discretization, and the correspondent

Author's email: anulovas@ipu.ru

solutions converge weakly as random processes. In the next two paragraphs we describe “cornerstones” of the proof.

3. Basing on (1)-(3), we can define a specular reflection operator acting on $f \in C_b^1$. Let us construct its ε-approximation. Hitting level $-\varepsilon$ at moment t, X^ε gets a jump $\Delta X_t^\varepsilon = \varepsilon$, and Y^ε — $\Delta Y_t^\varepsilon = 2|\Delta Y_{t-}^\varepsilon|$.

Lemma 2. *Let* $t > 0$, $\Delta\varphi_t^\varepsilon > 0$. *Then* $f'(t) < 0$, $\varphi_{t-}^\varepsilon < |f'(t)|$, *and* $\varphi_t^\varepsilon < 3|f'(t)|$.

Proof. $\Delta\varphi_t^\varepsilon > 0$ implies $Y_{t-}^\varepsilon < 0$, so $f'(t) + \varphi_{t-}^\varepsilon < 0$, which yields: $f'(t) < 0$; $|Y_{t-}^\varepsilon| + \varphi_{t-}^\varepsilon = |f'(t)|$ and $|Y_{t-}^\varepsilon| \le |f'(t)|$. Finally, $\Delta\varphi_t^\varepsilon = 2|Y_{t-}^\varepsilon| \le 2|f'(t)|$. □
Now take $\delta > 0$ and construct a partition: $t_0 = 0, t_1 =$ the first moment X^ε hits $-\varepsilon$, ..., $t_{i+1} =$ the first moment X^ε hits $-\varepsilon$ after $t_i + \delta$,... This partition assures: modulus of continuity of φ is “majorized” by that of f'. Indeed, let us shift the problem to $[t_i, \infty)$ and the function

$$X_{t_i}^\varepsilon + \int_{t_i}^t (\varphi_{t_i} + f'(s))ds.$$

The behavior of Y^ε on $[t_i, t_{i+1})$ may be regarded as an ε-approximation of $(f' + \varphi_{t_i})$. Applying then Lemma 2 we get $\varphi_{t_{i+1}} - \varphi_{t_i} \le -(f' + \varphi_{t_i})(t_i)$. Since $f' + \varphi_{t_{i+1}} \ge 0$, the right hand side is majorized by

$$\operatorname*{osc}_{[t_i, t_{i+1})} (f' + \varphi_{t_i}) = \operatorname*{osc}_{[t_i, t_{i+1})} f'.$$

4. Operator ψ^ε is an approximation of the Skorokhod reflection for function $X^\varepsilon - \psi^\varepsilon$. The difference of this function and f is non-decreasing. This allows to “majorize” ψ^ε by the Skorokhod reflection for f (which is equal to $-(\min_{s\in[0,t)} f(s) \wedge 0)$); see Theorem 1 [4]. And if the value of the solution of (1)-(3) at moment t ($X_t = 0, Y_{t-} < 0$), then the jump $\Delta\varphi_t$ is isolated (on the time axis). But as a continuous function ψ can’t grow at an isolated point.

Acknowledgements. Partially supported by RFBR grant 11-01-00949a.

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