Optimal dividend-payout in random discrete time

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1. The identification of optimal pay-out schemes for dividends to shareholders in an insurance context is a classical problem of risk theory. Given a stochastic process describing the surplus of an insurance portfolio as a function of time, it is a natural question at which points in time and to which amount dividends should be paid out to the shareholders. These pay-outs then reduce the current surplus. A popular optimality criterion is to maximize the expected total sum of discounted dividend payments until ruin (i.e. the dividend payments stop as soon as the surplus becomes negative for the first time). This problem was studied over the last decades under increasingly general model assumptions. Extending earlier work of de Finetti [4], Gerber [5] showed that if the surplus process is modeled by a random walk in discrete state space, then a so-called band-policy maximizes the expected sum of discounted dividend payments until ruin. He then also established this result for a continuous-time surplus process of compound Poisson type with downward jumps, and showed that in case of exponentially distributed claim sizes this optimal band-policy collapses to a barrier-policy, i.e. whenever the surplus process is above a certain barrier b, the excess is paid out as dividends immediately, and no dividends are paid out below this level b. In recent years, this problem was studied for general spectrally negative Lévy processes, and the most general conditions on such a process for which barrier-policies are optimal have recently been given in Loeffen & Renaud [7]. We refer to Schmidli [8] and Albrecher & Thonhauser [2] for an overview of mathematical tools and results in this area.

2. The implementation of the optimal pay-out policies that were identified for the above-mentioned continuous-time models of the surplus process need continuous observation of (and usually continuous intervention into) the surplus process, which can not be realized in practice. In this talk we therefore follow a somewhat different approach, namely to still consider a continuous-time model for the surplus process, as the latter is useful for many reasons, but to assume that observations (of possible ruin) and interventions (i.e. dividend pay-outs) are only possible at discrete points in time, and these time points are determined by a renewal process which is independent of the surplus process. This will enable a general treatment of the stochastic control problem to determine the optimal dividend pay-out scheme.

3. We will work with a general Lévy process (S_t) for the underlying surplus process. At (random) discrete time points $0 = Z_0 < Z_1 < \dots$ we are allowed to pay out dividends. We assume that the time lengths $T_n := Z_n - Z_{n-1}, n = 1, 2, \dots$ between interventions form a sequence of i.i.d. random variables which is also independent

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of the stochastic process (S_t). Thus it is enough to observe the process $\big(S(Z_n)\big)$ which evolves in discrete time. All quantities are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows we denote by

$$Y_n := S(Z_n) - S(Z_{n-1}), \quad n = 1, 2, \ldots$$

the increments of the surplus process. The aim is now to find a dividend pay-out policy such that the expected discounted dividends until ruin are maximized. Note that ruin is defined as the event that the surplus process at an observation time point is negative, so we disregard what happens between the time points (Z_n). Obviously the bivariate sequence (T_n, Y_n) is i.i.d.

In order to solve this problem we use the theory of Markov Decision Processes (for details see e.g. Bäuerle & Rieder [3]). More precisely we assume that $\mathbb{R}_+$ is the state space of the problem where the state x represents the current surplus. The action space is $\mathbb{R}_+$ where the action a represents the amount of money which is paid out as dividend. When the surplus is x we obtain the constraint that we have to restrict the dividend pay-out to the set $D(x) := [0, x]$. The one-stage reward of the problem is $r(x, a) =: a$. A dividend policy $\pi = (f_0, f_1, \ldots)$ is simply a sequence of decision rules f_n, where a function $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ is called decision rule when it is measurable and $f(x) \in D(x)$ is satisfied. The controlled surplus process (X_n) is given by the transition

$$X_n := X_{n-1} - f_{n-1}(X_{n-1}) + Y_n, \quad n = 1, 2, \ldots.$$

When we denote by

$$\tau := \inf\{n \in \mathbb{N}_0 : X_n < 0\}$$

the ruin time point in discrete time and by $\delta > 0$ the discount rate, then the expected discounted dividends under pay-out policy $\pi = (f_0, f_1, \ldots)$ are given by

$$V(x; \pi) := \mathbb{E}_x \Big[\sum_{n=0}^{\tau-1} e^{-\delta Z_n} f_n(X_n) \Big], \quad x \in \mathbb{R}_+.$$

The optimization problem then is

$$V(x) := \sup_{\pi} V(x; \pi), \quad x \in \mathbb{R}_+,$$

where the supremum is taken over all policies.

Under some mild assumptions it can be shown that the optimal policy is stationary and a *band policy*, i.e. the optimal policy is given by f^∞ and there exists a partition of $\mathbb{R}_+$ of the form $A \cup B = \mathbb{R}_+$ with

$$f(x) = \begin{cases} 0, & \text{if } x \in B, \\ x - z \text{ where } z = \sup\{y \,|\, y \in B \wedge 0 \le y < x\}, & \text{if } x \in A. \end{cases}$$

4. Suppose now that the surplus process is a compound Poisson process with claim arrival process (N_t) having intensity $\lambda > 0$ and exponentially distributed claim sizes U_i with parameter $\nu > 0$. The premium rate is again denoted by c. Thus we obtain

$$S_t = x + ct - \sum_{i=1}^{N_t} U_i, \quad t \geq 0.$$

The inter-observation times are also assumed to be exponentially distributed with parameter $\gamma > 0$ (i.e. the observations are determined by a homogeneous Poisson process with intensity γ).

In this case it can be shown that the optimal policy is stationary and a *barrier policy*, i.e. there exists a number $c \geq 0$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \leq c \\ x - c, & \text{if } x > c. \end{cases}$$

5. Consider now a sequence of the exponential models studied in the previous section. More precisely, let us assume that in the n-th model, the Poisson process (N_t^n) has intensity $\lambda_n := \lambda n$, the claim sizes U_i^n are exponentially distributed with parameter $\nu_n := \nu\sqrt{n}$ and the premium rate is $c_n := \frac{\lambda}{\nu}\sqrt{n}(\rho_n + 1)$ with $\lim_{n\to\infty}\sqrt{n}\rho_n = \kappa$. The parameter γ of the random observation time and the discount factor δ are kept fixed. Then it is well known that the corresponding compound Poisson process can be written as

$$\begin{aligned} S_t^n &:= x + c_n t - \sum_{i=1}^{N_t^n} U_i^n \\ &\stackrel{d}{=} x + \frac{\lambda}{\nu}\sqrt{n}(\rho_n + 1)t - \sum_{i=1}^{N_{nt}} \frac{U_i}{\sqrt{n}} \\ &= x + \frac{\lambda}{\nu}\sqrt{n}\rho_n t - \sqrt{2\lambda/\nu^2}\Big(\frac{\bar{S}(nt) - (\lambda/\nu)nt}{\sqrt{2\lambda/\nu^2}\sqrt{n}}\Big) \end{aligned}$$

where $\bar{S}(t) := \sum_{i=1}^{N_t} U_i$. From this representation it follows that (S_t^n) converges for $n \to \infty$ weakly to a diffusion (see e.g. Grandell [6, Sec.1.2]). More precisely we have

$$(S_t^n) \Rightarrow (x + \frac{\lambda}{\nu}\kappa\, t + \sqrt{2\lambda/\nu^2}W_t)$$

where $\Rightarrow$ denotes weak convergence on the space of càdlàg functions and (W_t) is a Brownian motion. Obviously the limiting model is again in the general Lévy class that we considered in the beginning. Since we know already from the previous section that for every exponential model a barrier-policy is optimal, one can show – by taking limits – that the same is true for a diffusion model.

6. The talk is based on [1].

References

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