# Stochastic Perron's method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games

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**1.** In [1], the authors introduce a stochastic version of Perron's method to construct viscosity (semi)-solutions for *linear* parabolic (or elliptic) equations, and use viscosity comparison as a substitute for verification (Itô's lemma). The present note extends the Stochastic Perron's method to the case of (double) obstacle problems associated to games of optimal stopping, the so called Dynkin games introduced in [2]. This is the first instance of a non-linear problem that can be treated using Stochastic Perron, and represents a very important step towards treating general stochastic control problems and their corresponding Hamilton-Jacobi-Bellman equations. As a matter of fact, we conjecture that basically any partial differential equation which is related to a stochastic representation can be potentially treated using some modification of what we call the Stochastic Perron's method. We intend to present some other important cases in future work.

**2. Overview of existing literature on (games of) optimal stopping.** Optimal stopping and the more general problem of optimal stopping games are fundamental problems in stochastic optimization. Such problems have been well studied for more than fifty year to various degrees of generality, and very general results have been obtained. If the optimal stopping is associated to Markov diffusions, there are two classic approaches to solve the problem:

1. The analytic approach consists in writing the Hamilton-Jacobi-Bellman equation (which takes the form of an obstacle problem), finding a smooth solution and then go over *verification arguments*. The method works only if the solution to the HJB is smooth enough to apply Itô's formula along the diffusion. This is particularly delicate if the diffusion degenerates.

2. The probabilistic approach consists in a very fine analysis of the value function(s), using heavily the Markov property and conditioning, to show a similar conclusion to the analytic approach: it is optimal to stop as soon as the player(s) reach(es) the contact region between the value function and the obstacle. In the case of optimal stopping (only one player) the value function can be characterized as the least excessive (super-harmonic) function. Recently, a similar characterization of the value function was studied for the case of games in [3]. Usually, the probabilistic approach is further used to draw other important conclusions, resembling the analytic approach. More precisely, it can be shown that the value function is a viscosity solution of the HJB. If a comparison results holds, then the value function is *the unique viscosity solution*, and finite-difference numerical methods can be used to approximate it.

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**3. Our contribution.** Compared to the existing large body of work on optimal stopping (games), we view our contribution as mostly conceptual. We provide here a new approach that lies *in between the analytic and the probabilistic approaches* described above. More precisely, we propose a *probabilistic version of the analytic approach.* We believe our method is novel in that

1. compared to the analytic approach, it does not require the existence of a smooth solution. This is because we do not apply Itô's formula to the solution of the PDE, but *only* to the smooth test functions.

2. compared to the probabilistic approach, we do not perform *any* direct analysis on the value function(s). As a matter of fact, even the very Markov property needed for such analysis is not assumed. The Markov property is hidden behind the uniqueness of the viscosity solution. This is all a consequence of the (same) fact the we apply Itô's lemma to the smooth test functions (as described above) along solutions of SDE without any Markov assumption on the SDE.

We believe our method displays a deeper connection between (stopped) diffusions and (viscosity solutions of) free boundary problems. The fine interplay between how much smoothness is needed for a solution of a PDE in order to apply Itô's formula along the SDE (which is needed in the classical analytic approach) is hidden behind the definition of *stochastic* super- and sub-solutions, which traces back to the seminal work of Stroock and Varadhan [6].

We could summarize the message of our main result as: if a viscosity comparison result for the HJB holds, then there is no need to either find a smooth solution of the HJB, or to analyze the value function(s) to solve the optimization problem. Formally, all classic results hold as expected, i.e., *the unique continuous (but possibly non-smooth) viscosity solution is equal to the value of the game and it is optimal for the player(s) to stop as soon as they reach their corresponding contact/stopping regions.* This amounts to a verification without smoothness, in the spirit of the analytic approach to optimal stopping. This resolution of the problem seems shorter (and more elementary) than the probabilistic approach described above. In addition, our main result tells us that the value function is equal to the infimum over stochastic super-solutions or the supremum over stochastic sub-solutions, resembling the probabilistic results in [3].

Compared to the previous work on Stochastic Perron's method [1], the contribution of the present note lies in *the precise and proper identification of stochastic sub- and super- solutions for the obstacle problem.* The technical contribution consists in proving that, having identified such a definition of stochastic solutions, the Perron's method actually does produce viscosity super- and sub-solutions. The proofs turn out to be very different from [1].

**Acknowledgements.** The research of E. Bayraktar was supported in part by the National Science Foundation under grants DMS 0906257, DMS 0955463, and DMS 1118673.

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