# Pricing of swing options in continuous time

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This talk is devoted to the pricing of swing options in continuous time. In general, the holder of a swing option has the right to exercise a certain total volume up to maturity, but she is subjected to some constraints. Depending on the formulation of the constraints, swing option pricing can be treated as a multiple stopping problem or as a stochastic control problem.

To be more precise, let us assume that the payoff of the option is modeled by an adapted stochastic process $X = (X(t),\ t \in [0, T])$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. We suppose that $X$ is nonnegative and the paths of $X$ are rightcontinuous with left limits. Moreover,

$$E[\sup_{0\le t\le T} |X(t)|^2] < \infty.$$

**1. Formulation as a multiple stopping problem.** Carmona and Touzi [2] suggested to formulate swing option pricing in continuous time via a multiple stopping problem. The holder of the option has the right to exercise the option up to $N$ times. We here use the convention that she receives $X(\tau)/N$, if she exercises the option at time $\tau$. So the aim of the holder of the option is to choose stopping times $\tau_1, \ldots, \tau_N$ in an 'admissible' way such that

$$E\left[\frac{1}{N}\sum_{\nu=0}^{N} X(\tau_\nu)\right]$$

is maximized. Here we already assume that the probability measure $P$ is a pricing measure rather than the physical measure, i.e. all tradable and storable assets are $\sigma$-martingales under $P$. It is industry practice to impose a minimal waiting time of $\delta > 0$ in between two exercises. E.g., when exercising a right involves the physical delivery of a commodity, this waiting time, which is known as the refraction period, is usually at least as large as the time required for delivery. Incorporating this constraint leads to the following multiple stopping problem: The price of the swing option with $N$ exercise rights and refraction period $\delta$ at a stopping time $\sigma$ is given by

$$Y^{*,N}(\sigma) := \operatorname*{esssup}_{(\tau_1,\ldots,\tau_N)\in\mathcal{S}^N_{\delta,\sigma}} \frac{1}{N}\sum_{\nu=1}^{N} E[X(\tau_\nu)|\mathcal{F}_\sigma], \tag{1}$$

where $\mathcal{S}^N_{\delta,\sigma}$ contains those $n$-tuples of stopping times $(\tau_1, \ldots, \tau_N)$ such that $\tau_1 \ge \sigma$ and $\tau_\nu \ge \tau_{\nu-1} + \delta$ for $\nu = 2, \ldots, N$. Here we apply the convention that $X(t) = 0$ for $t > T$, i.e. a right used later than $T$ remains in fact unexercised.

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In this setting we show the following reduction principle to a single stopping problem, which generalizes a result by Carmona and Touzi [2]:

There is an adapted process $X^N(t)$, whose discontinuities from the right are included in the set $D_N := \{T - \nu\delta;\ \nu = 0, \dots, N-1\}$, such that

$$Y^{*,N}(\sigma) = \operatorname*{esssup}_{\tau \in \mathcal{S}_\sigma} \frac{1}{N} E[X^N(\tau)|\mathcal{F}_\sigma] \tag{2}$$

and

$$E[\sup_{0 \le t < \infty} |X^N(t)|^2] < \infty.$$

Here $X^N(t)$ is a modification of $X(t) + (N-1)E[Y^{*,N-1}(t+\delta)|\mathcal{F}_t]$ such that

$$X^N(\tau) = X(\tau) + (N-1)E[Y^{*,N-1}(\tau+\delta)|\mathcal{F}_\tau]$$

for every stopping time $\tau$.

Some technical problems arise in the derivation of this result due to the fact that the Snell envelopes $Y^{*,N}(t)$ may exhibit discontinuities from the right for $N \ge 2$.

We also discuss existence of optimal families of stopping times under the additional assumption that $X$ is leftcontinuous in expectation. Moreover we derive a dual representation for the multiple stopping problem as a minimization problem over martingales and processes of bounded variation, which generalizes a result in discrete time by Schoenmakers [4].

**2. Formulation as optimal control problem.** Suppose now that the number of exercise rights tends to infinity and the refraction period $\delta_N$ tends to zero. If $\lim_{N\to\infty} N\delta_N = \frac{1}{L}$, then the natural limiting problem of (1) is the classical control problem

$$J(\sigma, Y) := \operatorname*{esssup}_{u \in U(\sigma,Y)} E\left[\int_\sigma^T u(s)X(s)ds \middle| \mathcal{F}_\tau\right] \tag{3}$$

where $U(\sigma, Y)$ is the set of all adapted processes with values in $[0, L]$ such that $\int_\sigma^T u(s)ds \le 1 - Y$. Here $u$ can be interpreted as the rate at which the volume is consumed by the holder of the option.

Passing to the limit $N \to \infty$ in (2) suggests that (an appropriate version of) $J(t, y)$ should solve the backward stochastic partial differential equation (BSPDE)

$$\begin{aligned} J(t,y) &= E\left[L\int_t^T (X(s) + D_y^+ J(s,y))_+ ds \middle| \mathcal{F}_t\right], \\ J(T,y) &= 0, \quad J(t,1) = 0. \end{aligned} \tag{4}$$

Here $D_y^+ J$ denotes the right-hand side derivative of $J$ in $y$. In a Markovian setting this BSPDE reduces formally to a classical Hamilton-Jacobi-Bellman equation. It can also be connected to stochastic Hamilton-Jacobi-Bellman equations in the sense of Peng [3]. We show that the value process $J(t, y)$ solves the BSPDE (4) and that the right-hand side derivative can be replaced by the left-hand side derivative in (4). To this end we discuss the connection of the discontinuities of $D_y^+ J(s,y)$ in time and space.

We also show that the derivative $D_y J(s,y)$ exists under the additional assumption that $X$ is leftcontinuous in expectation and represent it as an optimal stopping problem of $X$ restricted to some subset of predictable stopping times.

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## References

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