Control and Nash games with mean field effect

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Mean field theory has raised a lot of interest in the recent years, see in particular Lasry-Lions [11], [12], [13], Gueant-Lasry-Lions [8], Huang-Caines-Malhamé [9], [10], Buckdahn-Li-Peng [7]. There are a lot of applications. In general the applications concern approximating an infinite number of players with common behavior by a representative agent. This agent has to solve a control problem perturbed by a field equation, representing in some way the behavior of the average infinite number of agents. The state equation is modified by the expected value of some functional on the state. We first review in this presentation the linear-quadratic case. This has the advantage of getting explicit solutions. In particular this leads to the study of Riccati equations. We discuss two approaches. One in which the agent considers the mean field term as external, and an equilibrium occurs when this mean field term coincides with the average of his/her own action. The problem reduces to a fixed point. In another one, the mean field is a functional of the state and therefore the agent can influence it by his/her own decision. When there is no control, there is no difference between the two approaches. However, with control the two approaches are not equivalent. In particular, the fixed point approach leads to non-symmetric Riccati equations, which have no control interpretation. They raise interesting mathematical problems of their own. For nonlinear nonquadratic problems, the approach which has been explored is the endogeneous one. The control of the representative agent can influence the mean field term, which is the average of the agents state. The Dynamic pogramming approach fails, because of the so called inconsistency effect. Fortunately, the stochastic maximum principle can be applied. The adjoint variables are solutions of stochastic backward differential equations, with mean field terms. In the approach of Lasry-Lions, the starting point is a Nash equilibrium game for a very large number of players. In principle, the problem can be treated by Dynamic Programming. The Bellman equation becomes a system of nonlinear partial differential equations, for which the techniques of [2] can be considered. When the number of players becomes infinite, and all of them are identical, then going to the limit, one obtains an Hamilton-Jacobi-Bellman equation, with mean field term. The mean field term is reminiscent of the coupling with other players, which existed before going to the limit. We compare the various approaches, and their interprations as control problems. In Lasry-Lions approach the limit is obtained thanks to ergodic theory, which means that the limit control problem is an ergodic control problem, with mean field effect.

There is a different and interesting approach which also leads to similar types of P.D.E with mean-field terms. The state equation is the Chapman-Kolmogorov

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equation, describing the probability measure of the state. It is the dual control problem. Then, the Bellman equation can be interpreted as a necessary condition of optimality for the dual problem. To generate mean-field terms, it is sufficient to consider objective functions which are not just linear in the probability measure, but more complex.

This approach has a different type of application. In the traditional stochastic control problem, the objective functional is the expected value of a cost depending on the trajectory. So it is linear in the probability measure. This type of functional leaves out many current considerations in control theory, namely situations where one wants to take into consideration not just the expected value, but also the variance. This case occurs often in Risk Management. Moreover, one may be interested by several functionals on the trajectory, even though one is satisfied with expected values. If one combines these various expected values in a single pay-off, one is lead naturally to mean-field problems. They are meaningful even without considering ergodic theory, i. e. long term behavior.

Anyway, in all the previous considerations, the averaging approach reduces an infinite number agent to a representative agent, who has a control problem to solve, with an external effect, representing the averaged impact of the infinite number of players. Of course, this framework relies on the assumption that the players behave in a similar way. By construction, it eliminates the situation of a remaining Nash equilibrium for a finite number of players, with mean field terms.

In most real problems of economics, there is not just one representative agent and a large community of identical players, which impact with a mean field term. There is the situation of several major players, and large communities.

So a natural question is to consider the problem of these major players. They know that they can influence the community, and they also compete with each other. So the issue is that of differential games, with mean field terms, and not of mean field equations arising from the limit of a Nash equilibrium for an infinite number of players.

One way to recover this system of nonlinear P.D.E. with mean field terms is to consider averaging 2 within groups. Each of them is composed of an homogeneous community, but different communities are not identical.

To recover the system of nonlinear P.D.E. it is easier to proceed with the dual problems as explained above. One can consider a differential game for state equations which are probability distributions of states, and evolve according to Chapman-Kolmogorov equations. One recovers nonlinear systems of P.D.E. with mean field terms, with a different motivation. Another interesting feature of this approach is that we do not need to consider an ergodic situation, as it is the case in the standard approach of mean field theory. In fact, considering strictly positive discounts is quite meaningful in our applications. This leads to systems of non-linear P.D.E. with mean field coupling terms, that we can study with a minimum set of assumptions. The ergodic case, when the discount vanishes, requires much stringent assumptions, as it is already the case when there is no mean field term. We refer to Bensoussan-Frehse [2], [4] and Bensoussan-Frehse-Vogelgesang [5], [6] for the situation without mean field term. Basically our set of assumptions remains valid and we have to incorporate additional assumptions to deal with the mean field terms.

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