Efficient hedging of options with robust convex loss functionals

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1. We study the problem of partial hedging a European option in an incomplete financial market, modeled through a semimartingale discounted price process $S = (S_t)_{t\in[0,T]}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$, $T < \infty$.

Let P_σ denote the set of probability measures P^* equivalent to P such that S is a sigma-martingale with respect to P^*. We assume that S satisfies the condition of NFLVR (no free lunch with vanishing risk). As in [1], it is equivalent to $P_\sigma \neq \varnothing$.

We model the discounted payoff of a European option with an $\mathcal{F}_T$-measurable, nonnegative random variable H and assume $C_0 := \sup_{\mathsf{P}^*\in P_\sigma} \mathsf{E}^{\mathsf{P}^*}[H] < \infty$.

Then for a given initial capital $0 \le c \le C_0$ we have the following optimization problem

$$u(c) = \inf_{\xi\in Adm} \sup_{\mathsf{Q}\in\mathcal{Q}} \big\{ \mathsf{E}^{\mathsf{Q}}[l(H - V_T^\xi)^+] - \gamma(\mathsf{Q}) \big\}, \tag{1}$$

where $\mathcal{Q}$ is a convex family of absolutely continuous probability measures with respect to P, $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex, nondecreasing loss function, $\gamma\colon \mathcal{Q} \to \mathbb{R}_+$ is a convex function, Adm is a class of all admissible hedging strategies and $V_t^\xi := c + \int_0^t \xi_s dS_s$ is the corresponding value process, $t \in [0,T]$.

2. Efficient hedging was introduced and solved in a general semimartingale model in continuous time in [4]. The authors used the expected loss function as a risk measure.

The two-steps method of [4] can be analogously applied to our case and provides that

$$u(c) = \inf_{V\in\mathbb{A}} \sup_{\mathsf{Q}\in\mathcal{Q}} \big\{ \mathsf{E}^{\mathsf{Q}}[l(H - V)] - \gamma(\mathsf{Q}) \big\}, \tag{2}$$

where $\mathbb{A} := \big\{ V \in \mathcal{F}_T \mid \mathsf{P}(0 \le V \le H) = 1 \text{ and } \sup_{\mathsf{P}^*\in P_\sigma} \mathsf{E}^{\mathsf{P}^*}[V] \le c \big\}$.

3. In the present paper we provide a dual characterization of the value function of this optimal problem.

To be more concrete, the following dual-representation formula holds:

$$u(c) = -\inf_{y\ge 0} \inf_{\eta\in\mathcal{D}, \mathsf{Q}\in\mathcal{Q}} \big\{ \mathsf{E}^{\mathsf{Q}}[\tilde{V}_H(y \frac{\eta}{Z^{\mathsf{Q}}})] + \gamma(\mathsf{Q}) + \mathsf{c}\mathsf{y} \big\}, \tag{3}$$

where $Z^{\mathsf{Q}} = d\mathsf{Q}/d\mathsf{P}$, $\quad \tilde{V}_{H(w)}(\lambda) = \sup_{0\le x\le H(w)} \big\{ -\lambda x - l(H(w) - x) \big\}$ and $\mathcal{D} := \big\{ \eta \in L_1^+ \mid \mathsf{E}^{\mathsf{P}} \eta V \le 1 \text{ for all } V \in \mathbb{A} \big\}$.

Moreover the infimum in (2) is attained.

4. The same result is obtained in [5], but the authors used additional assumptions to prove it. In this paper it is shown that one can prove some facts by using

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the convex analysis methods and do not require additional assumptions such as conditions on differentiability of the loss function in [5].

In particular, we widely use a new approach to the notion of the f-divergence [2,3] which extends the domain of its definition to bounded finitely additive set functions taking nonnegative values.

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