On estimate for variational inequality associated to optimal stopping

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1. We consider a probability space $(\Omega, \mathcal{F}, P)$ and an n-dimensional Wiener process $w_t = (w_t^1, \dots, w_t^n)$ on it. Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary ($\partial D \in C^2$). Denote $\sigma(D) = \inf\{t \geq 0 : \ w_t \overline{\in} D\}$, and let $g = g(x)$, $c = c(x)$ be continuous functions defined on $\overline{D}$. Denote also by P_x the probability measure corresponding to the initial condition $w_0(\omega) = x$ and define the following optimal stopping problem

$$S(x) = \sup_{\tau \in \mathfrak{M}} E_x \biggl(g(w_\tau) I_{(\tau < \sigma(D))} + \int\limits_0^{\tau \wedge \sigma(D)} c(w_s)\, ds \biggr), \tag{1}$$

where $\mathfrak{M}$ is the class of all stopping times with respect to the filtration $F^w = (\mathcal{F}_t^w)_{t \geq 0}$. The optimal stopping problem consists in finding a payoff $S(x)$ and in defining the optimal stopping time τ^* at with the supremum (1) is achieved [1].

2. Denote by $H^1(D)$ the first order Sobolev space of functions $v = v(x)$ on D and let $H_0^1(D)$ be the subspace of $H^1(D)$ consisting of the functions $v = v(x)$, "equal to zero" on the boundary ∂D. Denote $K = \{v : \ v \in H_0^1(D), \ v(x) \geq g(x)\}$ and let $a(u, v)$ be scalar product in $H_0^1(D)$. The variational inequality is formulated as follows: find a function $u(x) \in K$ such that the inequality

$$a(u, v - u) \geq \int\limits_D c(x)(v(x) - u(x))\, dx \tag{2}$$

is fulfilled for any function $v(x) \in K$. In [2] A. Bensoussan has established the fundamental connection between the optimal stopping problem and the corresponding variational inequality. In particular, he has shown that

$$u(x) = S(x), \quad x \in \overline{D}, \tag{3}$$

and the estimate

$$\sup_{x \in \overline{D}} |u^2(x) - u^1(x)| \leq \sup_{x \in \overline{D}} |g^2(x) - g^1(x)| \tag{4}$$

holds, where the functions $u^i(x)$, $i = 1, 2$, represent the solutions of the variational inequality (2) for the functions $g^i(x)$, $i = 1, 2$.

3. Via the stochastic analysis the present paper gives an answer to the following question: does the uniform closeness of the functions $g^1(s)$ and $g^2(x)$ imply in a

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certain sense the closeness of the partial derivatives $\frac{\partial u^1(x)}{\partial x_i}$, $\frac{\partial u^2(x)}{\partial x_i}$, $i = 1, \dots, n$, of the corresponding solutions $u^1(x)$ and $u^2(x)$ of the variational inequality (2)? Using the results from [1] and [3] , we obtain a new estimate which is formulated as follows.

Let $g^i(x)$, $c^i(x)$, $i = 1, 2$, be two initial pairs of the variational inequality (2). Than for the solutions $u^i(x)$, $i = 1, 2$, of the problem (2) the global estimate

$$\int\limits_D d^2(x, \partial D) \big|\mathrm{grad}\,(u^2 - u^1)(x)\big|^2\, dx + \int\limits_D (u^2(x) - u^1(x))^2\, dx \leq$$
$$\leq C\Big[\Big(\sup_{x\in D} |g^2(x) - g^1(x)| + \sup_{x\in D} |c^2(x) - c^1(x)|\Big)\times$$
$$\times\Big(\sup_{x\in D} |g^1(x)| + \sup_{x\in D} |c^1(x)| + \sup_{x\in D} |g^2(x)| + \sup_{x\in D} |c^2(x)|\Big)\Big]$$

is valid, where $d(x, \partial D)$ is the distance from the point x to the boundary ∂D, C is a constant depending on the dimension of the space $\mathbb{R}^n$ and on the Lebesgue measure of D, i.e. $C = C(n, mes(D))$.

In [4] analogous estimates are used for optimal portfolio in the pricing problem of American type options.

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