On superhedging prices of contingent claims

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1. We consider a model of security market which consists of \(d+1\) assets, one bond and \(d\) stocks. We suppose that the price of the bond is constant and denote by \(S = (S^i)_{1 \leq i \leq d}\) the price process of the \(d\) stocks. The process \(S\) is assumed to be a semimartingale on a given filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)\). We assume that \(T > 0\) is a finite time horizon and \(\mathcal{F}_0\) is trivial, \(\mathcal{F}_T = \mathcal{F}\).

Consider an investor on our financial market. A (self-financing) portfolio \(\Pi\) of the investor is a pair \((x, H)\), where the constant \(x\) is the initial value of the portfolio and \(H = (H^i)_{1 \leq i \leq d}\) is a trading strategy of the investor, i.e. is a predictable \(S\)-integrable process specifying the amount of each asset held in the portfolio. The value process \(X = (X_t)_{0 \leq t \leq T}\) of such a portfolio \(\Pi\) is given by

\[
X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \leq t \leq T.
\]

We denote by \(\mathcal{H}\) the set of admissible trading strategies of the investor and by \(\mathcal{X}(x)\) the family of wealth processes with non-negative capital at any instant, i.e. \(X\) is of the form (1), \(X_t \geq 0\) for all \(t \in [0, T]\), and with initial value equal to \(x\).

2. Given a contingent claim \(B\) with maturity \(T\), we consider the following two values:

\[
\mathcal{V}_{\mathcal{H}}(B) = \inf\{x \in \mathbb{R} : \exists H \in \mathcal{H} : x + (H \cdot S)_T \geq B\}
\]

and

\[
\mathcal{V}_+(B) = \inf\{x \in \mathbb{R} : \exists X \in \mathcal{X}(x) : X_T \geq B\}.
\]

The values \(\mathcal{V}_{\mathcal{H}}(B)\) and \(\mathcal{V}_+(B)\) are called the superhedging prices of the claim \(B\) and are the smallest initial endowments that allow the investor to super-replicate \(B\) at maturity. But in the first case, investor is allowed to use trading strategies from the set \(\mathcal{H}\), and in the second case, the wealth process of the investor has to be non-negative.

3. Superhedging was introduced and investigated first by El Karoui and Quenez [1] in a continuous-time setting where the risky assets follow a multidimensional diffusion process. Delbaen and Schachermayer [2, 3] generalized these results to, respectively, a locally bounded and unbounded semimartingale model, under the \((NFLVR)\) condition. Theorem 1 extends the results of papers [2, 3]. In Theorem 2 we prove a new representation of the price \(\mathcal{V}_+(B)\) via the sets \(\mathcal{X}^s\) and \(\mathcal{X}^\sigma\) of supermartingale and \(\sigma\)-martingale densities respectively (see [4]).

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4. Let us firstly introduce some basic objects we need to formulate our main results, Theorems 1 and 2. Denote by $\mathcal{A}$ the following set: $\mathcal{A} = \{(H \cdot S)_T, H \in \mathcal{H}\}$. Let $\psi = 1 + |B|$. Then we construct the sets $\mathcal{C}^\psi$ and $\mathcal{R}$ by the following rules:

$$\mathcal{C}^\psi = (\mathcal{A} - L^0_+) \cap (\psi L^\infty),$$

$$\mathcal{R} = \left\{ \mu \in ba_+: \mu \left( \frac{1}{\psi} \right) = 1, \mu(\xi) \leq 0 \ \forall \xi \in \mathcal{C}^\psi \psi \right\}.$$

The elements of $\mathcal{R}$ are usually called separating measures in the literature, analogous to the concept of martingale measure, but in a more general setting.

**Theorem 1.** Assume that the set $\mathcal{H}$ is a convex cone, $\mathcal{V}_\mathcal{H}(\psi) < \infty$ and $\frac{B}{\psi} \in L^\infty$. Then

$$\mathcal{V}_\mathcal{H}(B) = \max_{\mu \in \mathcal{R}} \mu \left( \frac{B}{\psi} \right). \quad (4)$$

If $B \in L^\infty$ and $\mathcal{H}$ is a set $\mathcal{H}^{bb}$ of bounded from below wealth processes, then, under the (NFLVR) condition, formula (4) is reduced to the result of paper [2]. We can also reduce our formula to the one from paper [3], but under the additional assumption $\mathcal{V}_\mathcal{H}(\psi) < \infty$, where $\mathcal{H}^{\psi}$ is a set of $\psi$-admissible trading strategies, introduced in [3]. In comparison with papers [2, 3], we use an abstract class of trading strategies, also we do not need any assumptions on arbitrage on financial market and the maximum in formula (4) is attained.

**Theorem 2.** Assume that $B \in L^0_+$. Then, under the (NUPBR) condition (see [4]),

$$\mathcal{V}_+(B) = \sup_{Z \in \mathcal{Z}} E BZ_T = \sup_{Z \in \mathcal{Z}} E BZ_T. \quad (5)$$

If $\mathcal{H} = \mathcal{H}^{bb}$, then, in general, $\mathcal{V}_\mathcal{H}(B) \leq \mathcal{V}_+(B)$. It is easy to prove that, under the (NFLVR) condition, $\mathcal{V}_\mathcal{H}(B) = \mathcal{V}_+(B)$. We give an example which shows that, under the weaker (NUPBR) condition, it is possible to have $-\infty < \mathcal{V}_\mathcal{H}(B) < \mathcal{V}_+(B) < +\infty$.

**Acknowledgements.** The author is grateful to Alexander A. Gushchin for helpful suggestions and comments.

**References**


