On superhedging prices of contingent claims

Ruslan V. Khasanov

Moscow State University, Moscow, Russia

1. We consider a model of security market which consists of $d+1$ assets, one bond and d stocks. We suppose that the price of the bond is constant and denote by $S = (S^i)_{1\le i\le d}$ the price process of the d stocks. The process S is assumed to be a semimartingale on a given filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathsf{P})$. We assume that $T > 0$ is a finite time horizon and $\mathscr{F}_0$ is trivial, $\mathscr{F}_T = \mathscr{F}$.

Consider an investor on our financial market. A (self-financing) portfolio Π of the investor is a pair (x, H), where the constant x is the initial value of the portfolio and $H = (H^i)_{1\le i\le d}$ is a trading strategy of the investor, i.e. is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X = (X_t)_{0\le t\le T}$ of such a portfolio Π is given by

$$X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \le t \le T. \tag{1}$$

We denote by $\mathscr{H}$ the set of admissible trading strategies of the investor and by $\mathscr{X}(x)$ the family of wealth processes with non-negative capital at any instant, i.e. X is of the form (1), $X_t \ge 0$ for all $t \in [0, T]$, and with initial value equal to x.

2. Given a contingent claim B with maturity T, we consider the following two values:

$$\mathscr{V}_{\mathscr{H}}(B) = \inf\{x \in \mathbb{R} : \exists H \in \mathscr{H} : x + (H \cdot S)_T \ge B\} \tag{2}$$

and

$$\mathscr{V}_+(B) = \inf\{x \in \mathbb{R} : \exists X \in \mathscr{X}(x) : X_T \ge B\}. \tag{3}$$

The values $V_{\mathscr{H}}(B)$ and $\mathscr{V}_+(B)$ are called the superhedging prices of the claim B and are the smallest initial endowments that allow the investor to super-replicate B at maturity. But in the first case, investor is allowed to use trading strategies from the set $\mathscr{H}$, and in the second case, the wealth process of the investor has to be non-negative.

3. Superhedging was introduced and investigated first by El Karoui and Quenez [1] in a continuous-time setting where the risky assets follow a multidimensional diffusion process. Delbaen and Schachermayer [2, 3] generalized these results to, respectively, a locally bounded and unbounded semimartingale model, under the $(NFLVR)$ condition. Theorem 1 extends the results of papers [2, 3]. In Theorem 2 we prove a new representation of the price $\mathscr{V}_+(B)$ via the sets $\mathscr{Z}^s$ and $\mathscr{Z}^\sigma$ of supermartingale and σ-martingale densities respectively (see [4]).

Author's email: rus1886@mail.ru

4. Let us firstly introduce some basic objects we need to formulate our main results, Theorems 1 and 2. Denote by $\mathscr{A}$ the following set: $\mathscr{A} = \{(H \cdot S)_T,\, H \in \mathscr{H}\}$. Let $\psi = 1 + |B|$. Then we construct the sets $\mathscr{C}^\psi$ and $\mathscr{R}$ by the following rules:

$$\mathscr{C}^\psi = (\mathscr{A} - L^0_+) \cap (\psi L^\infty),$$

$$\mathscr{R} = \left\{ \mu \in ba_+ : \mu\left(\frac{1}{\psi}\right) = 1,\ \mu(\xi) \leqslant 0\ \forall \xi \in \frac{\mathscr{C}^\psi}{\psi} \right\}.$$

The elements of $\mathscr{R}$ are usually called separating measures in the literature, analogous to the concept of martingale measure, but in a more general setting.

Theorem 1. Assume that the set $\mathscr{H}$ is a convex cone, $\mathscr{V}_{\mathscr{H}}(\psi) < \infty$ and $\frac{B}{\psi} \in L^\infty$. Then

$$\mathscr{V}_{\mathscr{H}}(B) = \max_{\mu \in \mathscr{R}} \mu\left(\frac{B}{\psi}\right). \tag{4}$$

If $B \in L^\infty$ and $\mathscr{H}$ is a set $\mathscr{H}^{bb}$ of bounded from below wealth processes, then, under the $(NFLVR)$ condition, formula (4) is reduced to the result of paper [2]. We can also reduce our formula to the one from paper [3], but under the additional assumption $\mathscr{V}_{\mathscr{H}^\psi}(\psi) < \infty$, where $\mathscr{H}^\psi$ is a set of ψ-admissible trading strategies, introduced in [3]. In comparison with papers [2, 3], we use an abstract class of trading strategies, also we do not need any assumptions on arbitrage on financial market and the maximum in formula (4) is attained.

Theorem 2. Assume that $B \in L^0_+$. Then, under the $(NUPBR)$ condition (see [4]),

$$\mathscr{V}_+(B) = \sup_{Z \in \mathscr{Z}^s} \mathsf{E}\, B Z_T = \sup_{Z \in \mathscr{Z}^\sigma} \mathsf{E}\, B Z_T. \tag{5}$$

If $\mathscr{H} = \mathscr{H}^{bb}$, then, in general, $\mathscr{V}_{\mathscr{H}}(B) \leq \mathscr{V}_+(B)$. It is easy to prove that, under the $(NFLVR)$ condition, $\mathscr{V}_{\mathscr{H}}(B) = \mathscr{V}_+(B)$. We give an example which shows that, under the weaker $(NUPBR)$ condition, it is possible to have $-\infty < \mathscr{V}_{\mathscr{H}}(B) < \mathscr{V}_+(B) < +\infty$.

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