Sharp inequalities for maximum of skew Brownian motion

Yaroslav A. Lyulko
Lomonosov Moscow State University, Moscow, Russia

Let \( W^\alpha = (W^\alpha_t)_{t \geq 0} \) be a skew Brownian motion with parameter \( \alpha \in [0, 1] \) which can be defined as a unique strong solution \( X = (X_t)_{t \geq 0} \) of stochastic equation

\[
X_t = X_0 + B_t + (2\alpha - 1)L^0_t(X),
\]

where \( L^0_t(X) \) is the local time at zero of \( X_t \). In the present work we obtain maximal inequalities for skew Brownian motion. These inequalities generalize well-known results concerning standard Brownian motion \( B = (B_t)_{t \geq 0} \) (case \( \alpha = 1/2 \)) and its modulus \( |B| = (|B_t|)_{t \geq 0} \) (case \( \alpha = 1 \)). Namely, the authors of [1], [2] established that for any Markov time \( \tau \in \mathcal{M} \)

\[
E\left( \max_{0 \leq t \leq \tau} B_t \right) \leq \sqrt{E\tau}, \quad E\left( \max_{0 \leq t \leq \tau} |B_t| \right) \leq \sqrt{2E\tau},
\]

where \( \mathcal{M} \) is the set of all Markov times \( \tau \) (with respect to the natural filtration of \( B \)) with \( E\tau < \infty \). The main result of our work is contained in the following theorem (see [3]).

**Theorem 1.** For any Markov time \( \tau \in \mathcal{M} \) and for any \( \alpha \in (0, 1) \) we have

\[
E\left( \max_{0 \leq t \leq \tau} W^\alpha_t \right) \leq M_\alpha \sqrt{E\tau}, \tag{1}
\]

where \( M_\alpha = \alpha(1 + A_\alpha)/(1 - \alpha) \) and \( A_\alpha \) is the unique solution of the equation

\[
A_\alpha e^{A_\alpha + 1} = \frac{1 - 2\alpha}{\alpha^2},
\]

such that \( A_\alpha > -1 \). Inequality (1) is "sharp," i.e. for each \( T \geq 0 \) there exists a stopping time \( \tau \) such that \( E\tau = T \) and

\[
E\left( \max_{0 \leq t \leq \tau} W^\alpha_t \right) = M_\alpha \sqrt{E\tau}.
\]

Author’s email: yaroslav.lyulko@gmail.com
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References


