Sharp inequalities for maximum of skew Brownian motion

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Let $W^\alpha = (W_t^\alpha)_{t\geq 0}$ be a skew Brownian motion with parameter $\alpha \in [0,1]$ which can be defined as a unique strong solution $X = (X_t)_{t\geq 0}$ of stochastic equation

$$X_t = X_0 + B_t + (2\alpha - 1)L_t^0(X),$$

where $L_t^0(X)$ is the local time at zero of X_t. In the present work we obtain maximal inequalities for skew Brownian motion. These inequalities generalize well-known results concerning standard Brownian motion $B = (B_t)_{t\geq 0}$ (case $\alpha = 1/2$) and its modulus $|B| = (|B_t|)_{t\geq 0}$ (case $\alpha = 1$). Namely, the authors of [1], [2] established that for any Markov time $\tau \in \mathfrak{M}$

$$\mathsf{E}(\max_{0\leq t\leq\tau} B_t) \leq \sqrt{\mathsf{E}\tau}, \quad \mathsf{E}(\max_{0\leq t\leq\tau} |B_t|) \leq \sqrt{2\mathsf{E}\tau},$$

where $\mathfrak{M}$ is the set of all Markov times τ (with respect to the natural filtration of B) with $\mathsf{E}\tau < \infty$. The main result of our work is contained in the following theorem (see [3]).

Theorem 1. *For any Markov time $\tau \in \mathfrak{M}$ and for any $\alpha \in (0,1)$ we have*

$$\mathsf{E}\left(\max_{0\leq t\leq\tau} W_t^\alpha\right) \leq M_\alpha\sqrt{\mathsf{E}\tau}, \tag{1}$$

where $M_\alpha = \alpha(1 + A_\alpha)/(1 - \alpha)$ and A_α is the unique solution of the equation

$$A_\alpha e^{A_\alpha + 1} = \frac{1 - 2\alpha}{\alpha^2},$$

such that $A_\alpha > -1$. Inequality (1) is "sharp," i.e. for each $T \geq 0$ there exists a stopping time τ such that $\mathsf{E}\tau = T$ and

$$\mathsf{E}\left(\max_{0\leq t\leq\tau} W_t^\alpha\right) = M_\alpha\sqrt{\mathsf{E}\tau}.$$

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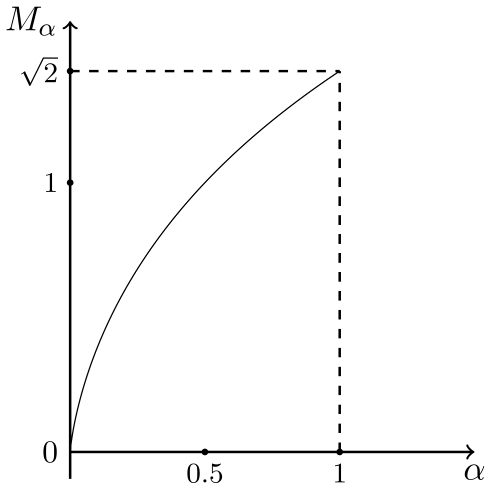

Fig. 1. The quantity M_α

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References

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