On a two-side disorder problem for a Brownian motion in a Bayesian setting

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1. Suppose we sequentially observe a stochastic process $X = (X_t)_{t\geq 0}$ having the structure

$$dX_t = \mu \mathbf{I}(t \geq \theta)dt + dB_t,$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, $\theta > 0$ and μ are *unobservable* random variables with known distributions, independent mutually and of B. The random variable θ is the moment when the drift of X_t changes its value from zero to μ, i.e. *"disorder"* happens.

In this paper we consider the case when random variables θ and μ have the following structure: θ takes value 0 with probability p ($q = 1 - p$ below) and it is exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$; μ takes values $\mu_1 < 0$ and $\mu_2 > 0$ with corresponding probabilities ρ_1 and $\rho_2 = 1 - \rho_1$. Being based upon the continuous observation of X our task is to detect the moment of disorder θ and define the value of μ (to test μ for hypotheses $H_1 : \mu = \mu_1$ and $H_2 : \mu = \mu_2$) with minimal loss.

For this, we consider *a sequential decision rule* $\delta = (\tau, d)$, where τ is a stopping time of the observed process X (with respect to the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$), and d is an $\mathcal{F}_\tau^X$-measurable random variable taking values d_1 and d_2. After stopping the observation at time τ the terminal decision d indicates which hypothesis on the drift value should be accepted: if $d = d_1$ we accept H_1 and if $d = d_2$ we accept H_2.

With each decision rule $\delta = (\tau, d)$ we associate the *Bayesian risk*

$$\mathbb{R}(\delta) = \mathbb{R}^\theta(\delta) + \mathbb{R}^\mu(\delta),$$

where

$$\mathbb{R}^\theta(\delta) = \mathsf{P}(\tau < \theta) + c\mathsf{E}[\tau - \theta]^+$$

is a combination of the probability of a "false alarm" and the average delay in detecting the "disorder" correctly, $c > 0$ is a given constant, and

$$\mathbb{R}^\mu(\delta) = a\mathsf{P}(d = d_1, \mu = \mu_2) + b\mathsf{P}(d = d_2, \mu = \mu_1)$$

is the average loss due to a wrong terminal decision, where $a > 0$ and $b > 0$ are given constants.

The problem then consists of finding the decision rule $\delta_* = (\tau_*, d_*)$ such that

$$\mathbb{R}(\delta_*) = \inf_\delta \mathbb{R}(\delta), \tag{1}$$

where the infimum is taken over all decision rules δ.

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Thus, the problem under consideration combines the classical problems of detecting the “disorder” and sequential hypothesis testing (for details see e.g. [1], Chapter VI).

2. Introduce the a posteriori probability processes $\pi^i = (\pi^i_t)_{t\geq 0}$, $i = 1, 2$ with

$$\pi^i_t = \mathsf{P}(\theta \leq t, \mu = \mu_i \,|\, \mathcal{F}^X_t), \quad i = 1, 2.$$

The method of solution of (1) is natural in such kind of problems and consists in reduction to an optimal stopping problem.

Theorem 1. *The 2-dimensional process $\pi = (\pi^1, \pi^2)$ is a Markov sufficient statistic in problem* (1)*. Moreover, the process π solves the following system of stochastic differential equations:*

$$d\pi^i_t = \lambda\rho_i(1 - \pi^1_t - \pi^2_t)dt + \pi^i_t \left[\frac{\mu_i}{\sigma} - \left(\frac{\mu_1}{\sigma}\pi^1_t + \frac{\mu_2}{\sigma}\pi^2_t\right)\right] d\overline{B}_t, \quad i = 1, 2,$$

where $\overline{B} = (\overline{B}_t)_{t\geq 0}$ is a Brownian motion (generally, different from B_t). The optimal stopping time τ_* can be found as the solution of the optimal stopping problem

$$V(\pi) = \inf_{\tau} \mathsf{E}_\pi \Bigg[1 - \pi^1_\tau - \pi^2_\tau + c\int_0^\tau (\pi^1_t + \pi^2_t)\, dt + a(\rho_1\pi^2_\tau + \rho_2(1 - \pi^1_\tau)) \wedge b(\rho_2\pi^1_\tau + \rho_1(1 - \pi^2_\tau))\Bigg], \quad (2)$$

where E_π denotes the mathematical expectation with respect to the measure P_π, under which π_t starts P_π-a.s. from the point π. Terminal decision function is defined as $d_* = d_1$ if $a(\rho_1\pi^2_\tau + \rho_2(1 - \pi^1_\tau)) < b(\rho_2\pi^1_\tau + \rho_1(1 - \pi^2_\tau))$ and $d_* = d_2$ otherwise.

In the talk we discuss analytical properties of the optimal stopping rules in the problem (2) and show how to compute optimal stopping boundary numerically.

Acknowledgements. This research was partially supported by the Russian Ministry of Education and Science through the project No. 14.740.11.1144.

References

[1] Peskir G., Shiryaev A. N. (2006). Optimal Stopping and Free-Boundary Problems. *Birkhäuser Verlag.*