On a two-side disorder problem for a Brownian motion in a Bayesian setting

Alexey Muravlev
Steklov Mathematical Institute, Moscow, Russia

1. Suppose we sequentially observe a stochastic process $X = (X_t)_{t \geq 0}$ having the structure

$$dX_t = \mu I(t \geq \theta)dt + dB_t,$$

where $B = (B_t)_{t \geq 0}$ is a standard Brownian motion, $\theta > 0$ and $\mu$ are unobservable random variables with known distributions, independent mutually and of $B$. The random variable $\theta$ is the moment when the drift of $X_t$ changes its value from zero to $\mu$, i.e. “disorder” happens.

In this paper we consider the case when random variables $\theta$ and $\mu$ have the following structure: $\theta$ takes value 0 with probability $p$ ($q = 1 - p$ below) and it is exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$; $\mu$ takes values $\mu_1 < 0$ and $\mu_2 > 0$ with corresponding probabilities $\rho_1$ and $\rho_2 = 1 - \rho_1$. Being based upon the continuous observation of $X$ our task is to detect the moment of disorder $\theta$ and define the value of $\mu$ (to test $\mu$ for hypotheses $H_1 : \mu = \mu_1$ and $H_2 : \mu = \mu_2$) with minimal loss.

For this, we consider a sequential decision rule $\delta = (\tau, d)$, where $\tau$ is a stopping time of the observed process $X$ (with respect to the natural filtration $(\mathcal{F}_t^X)_{t \geq 0}$), and $d$ is an $\mathcal{F}_\tau^X$-measurable random variable taking values $d_1$ and $d_2$. After stopping the observation at time $\tau$ the terminal decision $d$ indicates which hypothesis on the drift value should be accepted: if $d = d_1$ we accept $H_1$ and if $d = d_2$ we accept $H_2$.

With each decision rule $\delta = (\tau, d)$ we associate the Bayesian risk

$$\mathbb{R}(\delta) = \mathbb{R}^\theta(\delta) + \mathbb{R}^\mu(\delta),$$

where

$$\mathbb{R}^\theta(\delta) = P(\tau < \theta) + cE[\tau - \theta]^+$$

is a combination of the probability of a “false alarm” and the average delay in detecting the “disorder” correctly, $c > 0$ is a given constant, and

$$\mathbb{R}^\mu(\delta) = aP(d = d_1, \mu = \mu_2) + bP(d = d_2, \mu = \mu_1)$$

is the average loss due to a wrong terminal decision, where $a > 0$ and $b > 0$ are given constants.

The problem then consists of finding the decision rule $\delta_* = (\tau_*, d_*)$ such that

$$\mathbb{R}(\delta_*) = \inf_\delta \mathbb{R}(\delta),$$

where the infimum is taken over all decision rules $\delta$.

Author’s email: almurav@mi.ras.ru
Thus, the problem under consideration combines the classical problems of detecting the “disorder” and sequential hypothesis testing (for details see e.g. [1], Chapter VI).

2. Introduce the a posteriori probability processes
\[ \pi^i_t = P(\theta \leq t, \mu = \mu_i \mid F^X_t), \quad i = 1, 2. \]

The method of solution of (1) is natural in such kind of problems and consists in reduction to an optimal stopping problem.

**Theorem 1.** The 2-dimensional process \( \pi = (\pi^1, \pi^2) \) is a Markov sufficient statistic in problem (1). Moreover, the process \( \pi \) solves the following system of stochastic differential equations:
\[
    d\pi^i_t = \lambda \rho_i (1 - \pi^1_t - \pi^2_t) dt + \pi^i_t \left[ \frac{\mu_i}{\sigma} - \left( \frac{\mu_1}{\sigma} \pi^1_t + \frac{\mu_2}{\sigma} \pi^2_t \right) \right] dB_t, \quad i = 1, 2,
\]
where \( B = (B_t)_{t \geq 0} \) is a Brownian motion (generally, different from \( B_t \)). The optimal stopping time \( \tau^* \) can be found as the solution of the optimal stopping problem
\[
    V(\pi) = \inf_{\tau} E_{\pi} \left[ 1 - \pi^1_\tau - \pi^2_\tau + c \int_0^\tau (\pi^1_t + \pi^2_t) dt \right. \\
    \left. + a(\rho_1 \pi^2_\tau + \rho_2 (1 - \pi^1_\tau)) \wedge b(\rho_2 \pi^1_\tau + \rho_1 (1 - \pi^2_\tau)) \right], \quad (2)
\]
where \( E_{\pi} \) denotes the mathematical expectation with respect to the measure \( P_{\pi} \), under which \( \pi_t \) starts \( P_{\pi} \)-a.s. from the point \( \pi \). Terminal decision function is defined as \( d_* = d_1 \) if \( a(\rho_1 \pi^2_\tau + \rho_2 (1 - \pi^1_\tau)) < b(\rho_2 \pi^1_\tau + \rho_1 (1 - \pi^2_\tau)) \) and \( d_* = d_2 \) otherwise.

In the talk we discuss analytical properties of the optimal stopping rules in the problem (2) and show how to compute optimal stopping boundary numerically.

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**References**