Symmetric integrals and stochastic analysis

Farit S. Nasyrov

Ufa State Aviation Technical University, Ufa, Russia

1. In this paper following [1] we consider a symmetric integral $\int_0^t f(s,X(s))*dX(s)$ with respect to an arbitrary continuous function $X(s)$. If $X(s)$ is a path of Brownian motion, then the symmetric integral coincides with the Stratonovich integral.

Let $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_{N_n}^{(n)} = t$ be a sequence of partitions such that $\lim_{n\to\infty}\max_k(t_k^{(n)} - t_{k-1}^{(n)}) \to 0$. The limit $\lim\limits_{n\to\infty}\int_0^t f(s,X^{(n)}(s))(X^{(n)})'(s)ds$ is called a symmetric integral and is denoted by $\int_0^t f(s,X(s))*dX(s)$. Here $X^{(n)}(s)$ denotes a broken line.

Suppose that for almost all u:

(a) $f(s,u)$, $s \in [0,t]$, is a right-continuous bounded variation function;

(b) the total variation $|f|(t,u)$ of the function $f(s,u)$, $s \in [0,t]$, is an integrable function;

(c) $\int_0^t \mathbf{1}(s : X(s) = u)|f|(ds,u) = 0$;

then there exists a symmetric integral $\int_0^t f(s,X(s)) * dX(s)$.

The symmetric integral $\int_0^t f(s,X(s)) * dX(s)$ has the following properties:

(i) Let assumptions (a) – (c) hold, then

$$\int_0^t f(s,X(s)) * dX(s) = \int_{X(0)}^{X(t)} f(t,u)du - \int_R \int_0^t \kappa(u,X(0),X(s))f(ds,u)du,$$

here $\kappa(u,a,b) = sign(b-a)\mathbf{1}(a \wedge b < v < a \vee b)$.

(ii) Suppose that $F(t,u)$ has continuous partial derivatives F'_t, F'_u; then

$$F(t,X(t)) - F(0,X(0)) = \int_0^t F_u(s,X(s)) * dX(s) + \int_0^t F_s(s,X(s))ds.$$

2. A scalar first-order pathwise differential equation in differential form is written as the following equation

$$d\xi_s = \sigma(s,X(s),\xi_s) * dX(s) + b(s,X(s),\xi_s)ds, \quad \xi_0 = \xi(0), \quad s \in [0,t_0]. \qquad (1)$$

Here the first term in the right-hand corresponds to a symmetric integral, and the second term corresponds to a Riemann integral. The function $\xi(s) = \phi(s,X(s))$ is called a solution if the following conditions hold:

(i) the function $\phi(s,v)$ has continuous partial derivatives $\varphi'_v(s,v)$, $\varphi''_{sv}(s,v)$;

(ii) the function $\xi(s) = \phi(s,X(s))$ satisfies (1).

Author's email: farsagit@yandex.ru

From now on we make the assumption: the continuous function $X(s)$ is almost nowhere differentiable. The existence of solution of pathwise equation can be guaranteed by the following theorem.

Theorem 1 *Suppose that the functions $\sigma(s,v,\phi)$, $\sigma'_s(s,v,\phi)$, $\sigma'_\phi(s,v,\phi)$, $b(s,v,\phi)$ jointly continuous; then the following conditions are equivalent:*

(i) there exist a solution $\xi(s)=\phi(s,X(s))$;

(ii) the function $\xi(s)=\phi(s,u)$, $\varphi(0,X(0))=\xi(0)$, for almost all s satisfies the condition
$\phi'_v(s,X(s))=\sigma(s,X(s),\phi(s,X(s)))$; $\phi'_s(s,X(s))=b(s,X(s),\phi(s,X(s)))$.

Theorem 2 *Let all assumptions of Theorem 1 hold. Suppose that the function $b'_\phi(s,v,\phi)$ is jointly continuous; then there exists a unique solution of equation (1).*

Remark 1 Let $\sigma(s,v,\phi)\neq 0$. Using Theorem 1, we obtain the following equations chain

$$\phi'_v(s,v)=\sigma(s,v,\phi);\quad \phi'_s(s,X(s))=b(s,X(s),\phi(s,X(s))).$$

To find a solution of (1), we need to find a solution of this chain of equations.

For example, suppose that $\xi_t-\xi_0=\int_0^t[a\xi_s+b]*dX(s)+\int_0^t[h\xi_s+g]ds$ is a linear pathwise equation with respect to the symmetric integral. From Remark 1 it follows that $\phi'_u(t,u)=a\phi(t,u)+b$, $\phi'_t(t,u)|_{u=X(t)}=h\phi(t,X(t))+g$, $\phi(0,X(0))=\xi_0$. Hence $\phi(t,u)=\frac{1}{a}\left(e^{u+C(t)}-b\right)$, where $C(s)$ is an arbitrary function. In order to find a function $C(s)$, it is necessary to solve the equation $\frac{1}{a}e^{X(t)+C(t)}C'(t)=\frac{h}{a}\left(e^{X(t)+C(t)}-b\right)+g$ with initial condition $\frac{1}{a}\left(e^{X(0)+C(0)}-b\right)=\xi_0$.

3. The results of section 2 can be extended to more complex equations.

(i) Consider the equation $\eta(t)-\eta(0)=\sum\limits_{k=1}^{d}\int\limits_0^t a_k(s,\eta(s))*dW_k(s)+\int\limits_0^t b(s,\eta(s))ds$, $t\in[0,T]$, where $(W_1(s),...,W_d(s))$ is a multi-dimensional Brownian motion. The solution of this equation must be sought in the form of $\eta(s)=\phi(s,W_1(s),...,W_d(s))$. To find $\eta(s)$, it is necessary to solve the equations chain

$$\phi'_{u_k}(s,W_1(s),...,W_{k-1}(s),u_k,W_{k+1}(s),...,W_d(s))=$$

$$=a_k(s,\phi(s,W_1(s),...,W_{k-1}(s),u_k,W_{k+1}(s),...,W_d(s)),\ k=1,...,d,$$

$$\phi'_s(s,W_1(s),...,W_d(s))=b(s,\phi'_s(s,W_1(s),...,W_d(s))).$$

(ii) Similarly, for the evolutional differential equation

$$u(t,x)-u(0,x)=\int_0^t F_1\left(s,x,X(s),u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)ds+$$

$$+\int_0^t F_2\left(s,x,X(s),u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)*dX(s),\quad (s,x)\in R^+\times R^n,$$

$k_1 + ... + k_n = k \leq m$, the solution is sought in the form of $u(s,x) = u(s,x,X(s))$. To find the solution of this equation, it is necessary to solve the equations chain

$$\frac{\partial}{\partial v}u(s,x,v) = F_2\left(s,x,v,u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)\Big|_{u=u(s,x,v)},$$

$$\frac{\partial}{\partial s}u(s,x,v)|_{v=X(s)} = F_1\left(s,x,X(s),u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)\Big|_{u=u(s,x,X(s))}.$$

Note that this method can be applied to solve the problem of nonlinear filtering of diffusion processes.

4. The linearization problem (see [1] for more details) of the stochastic ordinary differential equations is to find a change of variables such that a transformed equation becomes a linear equation.

Theorem 3 *Suppose that the coefficients σ and b of the equation (1) are continuous and $\sigma \neq 0$. Then (1) is reducible to the linear differential equation $d\eta_t = A(t)\eta_t * dX(t) + B(t)\eta_t dt$.*

References

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[2] Nasyrov F. S. (2011). *Local times, symmetric integrals and stochastic analysis.* – Moscow: FIZMATLIT, (in Russian).