Multilevel primal and dual approaches for pricing American options

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1. Let $(Z_j)_{j\geq 0}$ be a nonnegative adapted process on a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_j)_{j\geq 0}, \mathbb{P})$ representing the discounted payoff of an American option, so that the holder of the option receives Z_j if the option is exercised at time $j \in \{0, \ldots, T\}$ with $T \in \mathbb{N}_+$. The pricing of American options can be formulated as a primal-dual problem. The primal representation corresponds to the following optimal stopping problems

$$Y_j^* := \max_{\tau \in \mathcal{T}[j,\ldots,T]} \mathbb{E}_{\mathcal{F}_j}[Z_\tau], \quad j = 0, \ldots, T,$$

where $\mathcal{T}[j, \ldots, T]$ is the set of $\mathbb{F}$-stopping times taking values in $\{j, \ldots, T\}$. The process $\left(Y_j^*\right)_{j\geq 0}$ is called the Snell envelope. Y^* is a supermartingale satisfying the Bellman principle

$$Y_j^* = \max\left(Z_j, \mathbb{E}_{\mathcal{F}_j}[Y_{j+1}^*]\right), \quad 0 \leq j < T, \quad Y_T^* = Z_T.$$

An exercise policy is a family of stopping times $(\tau_j)_{j=0,\ldots,T}$ such that $\tau_j \in \mathcal{T}[j, \ldots, T]$.

During the nineties the primal approach was the only method available. Some years later a quite different "dual" approach has been discovered by [8] and [5]. The next theorem summarizes their results.

Theorem 1. *Let $\mathcal{M}$ denote the space of adapted martingales, then we have the following dual representation for the value process Y_j^**

$$\begin{aligned} Y_j^* &= \inf_{\pi \in \mathcal{M}} \mathbb{E}_{\mathcal{F}_j}\left[\max_{s\in\{j,\ldots,T\}} (Z_s - \pi_s + \pi_j)\right] \\ &= \max_{s\in\{j,\ldots,T\}} (Z_s - \pi_s^* + \pi_j^*) \quad a.s., \end{aligned}$$

where

$$Y_j^* = Y_0^* + \pi_j^* - A_j^* \tag{1}$$

is the (unique) Doob decomposition of the supermartingale Y_j^*. That is, π^* is a martingale and A^* is an increasing process with $\pi_0 = A_0 = 0$ such that (1) holds.

2. Assume that we are given a stopping family (τ_j) that is *consistent,* i.e.

$$\tau_j > j \Rightarrow \tau_j = \tau_{j+1}, \quad j = 0, \ldots, T-1.$$

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The stopping policy defines a lower bound for Y^* via

$$Y_j = \mathbb{E}_{\mathcal{F}_j}[Z_{\tau_j}], \quad j = 0, \ldots, T.$$

Consider now a new family $(\widehat{\tau}_j)_{j=0,\ldots,T}$ defined by

$$\widehat{\tau}_j := \inf\left\{k : j \le k < T,\ Z_k \ge \mathbb{E}_{\mathcal{F}_k}[Z_{\tau_{k+1}}]\right\} \wedge T. \tag{2}$$

The basic idea behind (2) goes back to [6] in fact. For more general versions of policy iteration and their analysis, see [7]. Next, we introduce the $(\mathcal{F}_j)$-martingale

$$\pi_j = \sum_{k=1}^{j} \left(\mathbb{E}_{\mathcal{F}_k}[Z_{\tau_k}] - \mathbb{E}_{\mathcal{F}_{k-1}}[Z_{\tau_k}]\right), \quad j = 0, \ldots, T, \tag{3}$$

and then consider,

$$\widetilde{Y}_j := \mathbb{E}_{\mathcal{F}_j}\left[\max_{k=j,\ldots,T}(Z_k - \pi_k + \pi_j)\right],$$

along with

$$\widehat{Y}_j := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_j}], \quad j = 0, \ldots, T.$$

The following theorem states that $\widehat{Y}$ is an improvement of Y and that the Snell envelope process Y_j^* lies between $\widehat{Y}_j$ and $\widetilde{Y}_j$ with probability 1.

Theorem 2. *It holds*

$$Y_j \le \widehat{Y}_j \le Y_j^* \le \widetilde{Y}_j, \quad j = 0, \ldots, T \quad a.s.$$

3. The main issue in the Monte Carlo construction of $\widehat{Y}$ and $\widetilde{Y}$ in a Markovian environment is the estimation of the conditional expectations in (2) and (3). We thus assume that the cash-flow Z_j is of the form $Z_j = Z_j(X_j)$ for some underlying (possibly high-dimensional) Markovian process X . A canonical approach is the use of sub simulations. In this respect we consider an enlarged probability space $(\Omega, \mathbb{F}', \mathbb{P})$, where $\mathbb{F}' = (\mathcal{F}'_j)_{j=0,\ldots,T}$ and $\mathcal{F}_j \subset \mathcal{F}'_j$ for each j. On the enlarged space we consider $\mathcal{F}'_j$ measurable estimations $\mathcal{C}_{j,M}$ of $C_j = \mathbb{E}_{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right]$ as being standard Monte Carlo estimates based on M sub simulations. More precisely

$$\mathcal{C}_{j,M} = \frac{1}{M}\sum_{m=1}^{M} Z_{\tau_{j+1}^{(m)}}(X^{j,X_j}_{\tau_{j+1}^{(m)}})$$

where the $\tau_{j+1}^{(m)}$ are evaluated on M sub trajectories all starting at time j in X_j. Obviously, $\mathcal{C}_{j,M}$ is an unbiased estimator for C_j with respect to $\mathbb{E}_{\mathcal{F}_j}[\cdot]$. We thus end up with a simulation based versions of (2) and (3) respectively,

$$\widehat{\tau}_{j,M} := \inf\{k : j \le k < T,\ Z_k > \mathcal{C}_{k,M}\} \wedge T, \quad j = 0, ..., T,$$

$$\pi_{j,M} \quad := \quad \sum_{k=1}^{j} (Z_k - \mathcal{C}_{k-1,M})\, 1_{\{\tau_k = k\}} + \sum_{k=1}^{j} (\mathcal{C}_{k,M} - \mathcal{C}_{k-1,M})\, 1_{\{\tau_k > k\}}.$$

Denote

$$\widehat{Y}_{j,M} := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_{j,M}}], \quad j = 0, \ldots, T$$

and

$$\widetilde{Y}_{j,M} := \mathbb{E}_{\mathcal{F}_j}\left[\max_{k=j,\ldots,T}(Z_k - \pi_{k,M} + \pi_{j,M})\right].$$

Theorem 3. *Let us assume that there exist constants* $B_{0,j} > 0$, $j = 0, \ldots, T-1$, *and* $\alpha > 0$, *such that for any* $\delta > 0$ *and* $j = 0, \ldots, T-1$,

$$\mathbb{P}(|\mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_{j+1}}] - Z_j| \leq \delta) \leq B_{0,j}\delta^{\alpha}.$$

Further suppose that there are constants B_1 *and* B_2, *such that* $|Z_j| < B_1$ *and*

$$Var_{\mathcal{F}_j}[Z_{\tau_{j+1}}] := \mathbb{E}_{\mathcal{F}_j}[(Z_{\tau_{j+1}} - C_j)^2] < B_2, \quad a.s. \tag{4}$$

for $j = 0, \ldots, T-1$. *It then holds,*

$$|\widehat{Y}_0 - \widehat{Y}_{0,M}| \leq M^{-\frac{1+\alpha}{2}}\, B \sum_{k=0}^{T-1} B_{0,k},$$

with some constant B *depending only on* α, B_1 *and* B_2. *Moreover, if for any* $\delta > 0$

$$\mathbb{P}(|\mathbb{E}_{\mathcal{F}_j}[Z_{\tau_{j+1}}] - Z_j| \leq \delta) \leq \overline{B}_{0,j}\delta^{\overline{\alpha}}$$

with some positive constants $\overline{\alpha}$ *and* $\overline{B}_{0,j}$, $j = 0, \ldots, T-1$, *then*

$$\mathbb{E}[(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0})^2] \leq M^{-\overline{\alpha}/2} 2B_1^2 \overline{B} \sum_{j=0}^{T-1} \overline{B}_{0,j}.$$

Theorem 4. *Introduce for* $\mathcal{Z} := \max_{j=0,\ldots,T}(Z_j - \pi_j)$, *the random set*

$$\mathcal{Q} = \{j : Z_j - \pi_j = \mathcal{Z}\},$$

and the $\mathcal{F}_T$ *measurable random variable*

$$\Lambda := \min_{j \notin \mathcal{Q}} (\mathcal{Z} - Z_j + \pi_j),$$

with $\Lambda := +\infty$ *if* $\mathcal{Q} = \{0, \ldots, T\}$. *Obviously* $\Lambda > 0$ *a.s. Further suppose that*

$$\mathbb{E}[\Lambda^{-\xi}] < \infty \text{ for some } 0 < \xi \leq 1, \quad \text{and} \quad \#\mathcal{Q} = 1.$$

It then holds,

$$\left|\widetilde{Y}_0 - \widetilde{Y}_{0,M}\right| \leq CM^{-\frac{\xi+1}{2}}$$

for some constant C.

For a fixed natural number L and a geometric sequence $m_l = m_0\kappa^l$, for some m_0, $\kappa \in \mathbb{N}$, $\kappa \geq 2$, we consider in the spirit of [4] the telescoping sum

$$\widehat{Y}_{m_L} = \widehat{Y}_{m_0} + \sum_{l=1}^{L}\left(\widehat{Y}_{m_l} - \widehat{Y}_{m_{l-1}}\right),$$

where $\widehat{Y}_m := \widehat{Y}_{0,m}$. Next we take a set of natural numbers $\mathbf{n} := (n_0, \ldots, n_L)$ satisfying $n_0 > ... > n_L \geq 1$, and simulate an initial set of cash-flows

$$\left\{Z^{(j)}_{\widehat{\tau}_{m_0}}, \quad j = 1, ..., n_0\right\},$$

due to an initial set of trajectories $\{X^{0,x,(j)}_{\cdot}, j = 1, ..., n_0\}$, where

$$Z^{(j)}_{\widehat{\tau}_{m_0}} := Z_{\widehat{\tau}^{(j)}_{0,m_0}}\left(X^{0,x,(j)}_{\widehat{\tau}^{(j)}_{0,m_0}}\right).$$

Next we simulate *independently* for each level $l = 1, ..., L$, a set of pairs

$$\left\{(Z^{(j)}_{\widehat{\tau}_{m_l}}, Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}), \quad j = 1, \ldots, n_l\right\}$$

due to a set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_l$, to obtain the multilevel estimator

$$\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0}\sum_{j=1}^{n_0} Z^{(j)}_{\widehat{\tau}_{m_0}} + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{j=1}^{n_l}\left(Z^{(j)}_{\widehat{\tau}_{m_l}} - Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}\right) \text{ for estimating } \widehat{Y}. \tag{5}$$

4. With the notations of the previous section we define

$$\widetilde{Y}_{m_L} = \widetilde{Y}_{m_0} + \sum_{l=1}^{L}[\widetilde{Y}_{m_l} - \widetilde{Y}_{m_{l-1}}],$$

where $\widetilde{Y}_m := \widetilde{Y}_{0,m}$. Given a sequence $\mathbf{n} = (n_0, \ldots, n_L)$ with $1 \leq n_0 < \ldots < n_L$, we then simulate for $l = 0$ an initial set of trajectories

$$\left\{(Z^{(i)}_j, \pi^{(i)}_{j,m_0}), \quad i = 1, ..., n_0, \quad j = 0, \ldots, T,\right\}$$

of the two-dimensional vector process (Z_j, π_{j,m_0}), and then for each level $l = 1, \ldots, L$, *independently,* a set of trajectories

$$\left\{(Z^{(i)}_j, \pi^{(i)}_{j,m_{l-1}}, \pi^{(i)}_{j,m_l}), \quad i = 1, ..., n_l, \quad j = 0, \ldots, T\right\}$$

of the vector process $(Z_j, \pi_{j,m_{l-1}}, \pi_{j,m_l})$. Based on this simulation we consider the following multilevel estimator:

$$\widetilde{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0}\sum_{i=1}^{n_0}\mathcal{Z}^{(i)}_{m_0} + \sum_{l=1}^{L}\frac{1}{n_l}\sum_{i=1}^{n_l}[\mathcal{Z}^{(i)}_{m_l} - \mathcal{Z}^{(i)}_{m_{l-1}}] \tag{6}$$

with $\mathcal{Z}_{m_l}^{(i)} := \max_{j=0,\dots,T}\left(Z_j^{(i)} - \pi_{j,m_l}^{(i)}\right)$, $i = 1,\dots,n_l$, $l = 0,\dots,L$.

5. We now consider the numerical complexity of the multilevel estimators (5) and (6), for convenience generically denoted by $X_{\mathbf{n},\mathbf{m}}$. Assume that there are some positive constants γ, β, μ_∞, σ_∞ and $\mathcal{V}_\infty$ such that $\mathrm{Var}[\mathcal{X}_m] \le \sigma_\infty^2$,

$$|X - \mathbb{E}[\mathcal{X}_m]| \le \mu_\infty m^{-\gamma}, \quad m \in \mathbb{N} \quad \text{and} \tag{7}$$

$$\mathbb{E}[\mathcal{X}_{m_l} - \mathcal{X}_{m_{l-1}}]^2 \le \mathcal{V}_\infty m_l^{-\beta}, \quad l = 1,\dots,L. \tag{8}$$

Theorem 5. *Let us assume that $0 < \beta \le 1$, $\gamma \ge \frac{1}{2}$ and $m_l = m_0\kappa^l$ for some fixed κ and $m_0 \in \mathbb{N}$. Fix some $0 < \epsilon < 1$. Let $L = L(\epsilon)$ be the integer part of*

$$\gamma^{-1}\ln^{-1}\kappa \ln\left[\frac{\sqrt{2}\mu_\infty}{m_0^\gamma \epsilon}\right], \quad \textit{and} \quad n_l = n_0\kappa^{-l(1+\beta)/2} \quad \textit{with}$$

$$n_0 = n_0(\epsilon) = \frac{2\sigma_\infty^2}{\epsilon^2} + \frac{2\mathcal{V}_\infty}{\epsilon^2 m_0^\beta}\frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}\kappa^{(1-\beta)/2}.$$

Then the complexity needed to achieve the accuracy $\varepsilon := \sqrt{\mathbb{E}[(X - X_{\mathbf{n},\mathbf{m}})^2]} < \varepsilon$ is

$$\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon) = O(\epsilon^{-2-\frac{1-\beta}{\gamma}}), \quad \epsilon \searrow 0, \quad \textit{for } \beta < 1,$$

$$\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon) = O(\epsilon^{-2}\ln^2\epsilon), \quad \epsilon \searrow 0, \quad \textit{for } \beta = 1.$$

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