International Conference

STOCHASTIC OPTIMIZATION and OPTIMAL STOPPING

BOOK OF ABSTRACTS

Moscow, 24-28 September 2012

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Preface

This book includes abstracts of the papers presented on International Conference "Stochastic Optimization and Optimal Stopping" held 24–28 September, 2012 in Steklov Mathematical Institute, Moscow, Russia. In total, there are 38 papers covering areas such as stochastic control, optimal stopping problems, stochastic games, changepoint detection problems, stochastic differential equations, and mathematical finance.

The Conference is organized by Steklov Mathematical Institute, Laboratory for Structural Methods of Data Analysis in Predictive Modeling, and Center for Structural Data Analysis and Optimization. The financial support of the conference is provided by the Government of the Russian Federation, grant ag.11.G34.31.0073.

We would like to thank all the participants, the members of the organizing and scientific committees for putting this conference together.

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Plenary talks

Optimal dividend-payout in random discrete time

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1. The identification of optimal pay-out schemes for dividends to shareholders in an insurance context is a classical problem of risk theory. Given a stochastic process describing the surplus of an insurance portfolio as a function of time, it is a natural question at which points in time and to which amount dividends should be paid out to the shareholders. These pay-outs then reduce the current surplus. A popular optimality criterion is to maximize the expected total sum of discounted dividend payments until ruin (i.e. the dividend payments stop as soon as the surplus becomes negative for the first time). This problem was studied over the last decades under increasingly general model assumptions. Extending earlier work of de Finetti [4], Gerber [5] showed that if the surplus process is modeled by a random walk in discrete state space, then a so-called band-policy maximizes the expected sum of discounted dividend payments until ruin. He then also established this result for a continuous-time surplus process of compound Poisson type with downward jumps, and showed that in case of exponentially distributed claim sizes this optimal band-policy collapses to a barrier-policy, i.e. whenever the surplus process is above a certain barrier b, the excess is paid out as dividends immediately, and no dividends are paid out below this level b. In recent years, this problem was studied for general spectrally negative Lévy processes, and the most general conditions on such a process for which barrier-policies are optimal have recently been given in Loeffen & Renaud [7]. We refer to Schmidli [8] and Albrecher & Thonhauser [2] for an overview of mathematical tools and results in this area.

2. The implementation of the optimal pay-out policies that were identified for the above-mentioned continuous-time models of the surplus process need continuous observation of (and usually continuous intervention into) the surplus process, which can not be realized in practice. In this talk we therefore follow a somewhat different approach, namely to still consider a continuous-time model for the surplus process, as the latter is useful for many reasons, but to assume that observations (of possible ruin) and interventions (i.e. dividend pay-outs) are only possible at discrete points in time, and these time points are determined by a renewal process which is independent of the surplus process. This will enable a general treatment of the stochastic control problem to determine the optimal dividend pay-out scheme.

3. We will work with a general Lévy process (S_t) for the underlying surplus process. At (random) discrete time points $0 = Z_0 < Z_1 < \dots$ we are allowed to pay out dividends. We assume that the time lengths $T_n := Z_n - Z_{n-1}, n = 1, 2, \dots$ between interventions form a sequence of i.i.d. random variables which is also independent

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of the stochastic process (S_t). Thus it is enough to observe the process $\big(S(Z_n)\big)$ which evolves in discrete time. All quantities are assumed to be defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In what follows we denote by

$$Y_n := S(Z_n) - S(Z_{n-1}), \quad n = 1, 2, \ldots$$

the increments of the surplus process. The aim is now to find a dividend pay-out policy such that the expected discounted dividends until ruin are maximized. Note that ruin is defined as the event that the surplus process at an observation time point is negative, so we disregard what happens between the time points (Z_n). Obviously the bivariate sequence (T_n, Y_n) is i.i.d.

In order to solve this problem we use the theory of Markov Decision Processes (for details see e.g. Bäuerle & Rieder [3]). More precisely we assume that $\mathbb{R}_+$ is the state space of the problem where the state x represents the current surplus. The action space is $\mathbb{R}_+$ where the action a represents the amount of money which is paid out as dividend. When the surplus is x we obtain the constraint that we have to restrict the dividend pay-out to the set $D(x) := [0, x]$. The one-stage reward of the problem is $r(x, a) =: a$. A dividend policy $\pi = (f_0, f_1, \ldots)$ is simply a sequence of decision rules f_n, where a function $f_n : \mathbb{R}_+ \to \mathbb{R}_+$ is called decision rule when it is measurable and $f(x) \in D(x)$ is satisfied. The controlled surplus process (X_n) is given by the transition

$$X_n := X_{n-1} - f_{n-1}(X_{n-1}) + Y_n, \quad n = 1, 2, \ldots.$$

When we denote by

$$\tau := \inf\{n \in \mathbb{N}_0 : X_n < 0\}$$

the ruin time point in discrete time and by $\delta > 0$ the discount rate, then the expected discounted dividends under pay-out policy $\pi = (f_0, f_1, \ldots)$ are given by

$$V(x; \pi) := \mathbb{E}_x \Big[\sum_{n=0}^{\tau-1} e^{-\delta Z_n} f_n(X_n) \Big], \quad x \in \mathbb{R}_+.$$

The optimization problem then is

$$V(x) := \sup_{\pi} V(x; \pi), \quad x \in \mathbb{R}_+,$$

where the supremum is taken over all policies.

Under some mild assumptions it can be shown that the optimal policy is stationary and a *band policy*, i.e. the optimal policy is given by f^∞ and there exists a partition of $\mathbb{R}_+$ of the form $A \cup B = \mathbb{R}_+$ with

$$f(x) = \begin{cases} 0, & \text{if } x \in B, \\ x - z \text{ where } z = \sup\{y \,|\, y \in B \wedge 0 \le y < x\}, & \text{if } x \in A. \end{cases}$$

4. Suppose now that the surplus process is a compound Poisson process with claim arrival process (N_t) having intensity $\lambda > 0$ and exponentially distributed claim sizes U_i with parameter $\nu > 0$. The premium rate is again denoted by c. Thus we obtain

$$S_t = x + ct - \sum_{i=1}^{N_t} U_i, \quad t \geq 0.$$

The inter-observation times are also assumed to be exponentially distributed with parameter $\gamma > 0$ (i.e. the observations are determined by a homogeneous Poisson process with intensity γ).

In this case it can be shown that the optimal policy is stationary and a *barrier policy*, i.e. there exists a number $c \geq 0$ such that

$$f(x) = \begin{cases} 0, & \text{if } x \leq c \\ x - c, & \text{if } x > c. \end{cases}$$

5. Consider now a sequence of the exponential models studied in the previous section. More precisely, let us assume that in the n-th model, the Poisson process (N_t^n) has intensity $\lambda_n := \lambda n$, the claim sizes U_i^n are exponentially distributed with parameter $\nu_n := \nu\sqrt{n}$ and the premium rate is $c_n := \frac{\lambda}{\nu}\sqrt{n}(\rho_n + 1)$ with $\lim_{n\to\infty} \sqrt{n}\rho_n = \kappa$. The parameter γ of the random observation time and the discount factor δ are kept fixed. Then it is well known that the corresponding compound Poisson process can be written as

$$\begin{aligned} S_t^n &:= x + c_n t - \sum_{i=1}^{N_t^n} U_i^n \\ &\overset{d}{=} x + \frac{\lambda}{\nu}\sqrt{n}(\rho_n + 1)t - \sum_{i=1}^{N_{nt}} \frac{U_i}{\sqrt{n}} \\ &= x + \frac{\lambda}{\nu}\sqrt{n}\rho_n t - \sqrt{2\lambda/\nu^2}\Big(\frac{\bar{S}(nt) - (\lambda/\nu)nt}{\sqrt{2\lambda/\nu^2}\sqrt{n}}\Big) \end{aligned}$$

where $\bar{S}(t) := \sum_{i=1}^{N_t} U_i$. From this representation it follows that (S_t^n) converges for $n \to \infty$ weakly to a diffusion (see e.g. Grandell [6, Sec.1.2]). More precisely we have

$$(S_t^n) \Rightarrow (x + \frac{\lambda}{\nu}\kappa\, t + \sqrt{2\lambda/\nu^2} W_t)$$

where $\Rightarrow$ denotes weak convergence on the space of càdlàg functions and (W_t) is a Brownian motion. Obviously the limiting model is again in the general Lévy class that we considered in the beginning. Since we know already from the previous section that for every exponential model a barrier-policy is optimal, one can show – by taking limits – that the same is true for a diffusion model.

6. The talk is based on [1].

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Stochastic Perron's method and verification without smoothness using viscosity comparison: obstacle problems and Dynkin games

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1. In [1], the authors introduce a stochastic version of Perron's method to construct viscosity (semi)-solutions for *linear* parabolic (or elliptic) equations, and use viscosity comparison as a substitute for verification (Itô's lemma). The present note extends the Stochastic Perron's method to the case of (double) obstacle problems associated to games of optimal stopping, the so called Dynkin games introduced in [2]. This is the first instance of a non-linear problem that can be treated using Stochastic Perron, and represents a very important step towards treating general stochastic control problems and their corresponding Hamilton-Jacobi-Bellman equations. As a matter of fact, we conjecture that basically any partial differential equation which is related to a stochastic representation can be potentially treated using some modification of what we call the Stochastic Perron's method. We intend to present some other important cases in future work.

2. Overview of existing literature on (games of) optimal stopping. Optimal stopping and the more general problem of optimal stopping games are fundamental problems in stochastic optimization. Such problems have been well studied for more than fifty year to various degrees of generality, and very general results have been obtained. If the optimal stopping is associated to Markov diffusions, there are two classic approaches to solve the problem:

1. The analytic approach consists in writing the Hamilton-Jacobi-Bellman equation (which takes the form of an obstacle problem), finding a smooth solution and then go over *verification arguments*. The method works only if the solution to the HJB is smooth enough to apply Itô's formula along the diffusion. This is particularly delicate if the diffusion degenerates.

2. The probabilistic approach consists in a very fine analysis of the value function(s), using heavily the Markov property and conditioning, to show a similar conclusion to the analytic approach: it is optimal to stop as soon as the player(s) reach(es) the contact region between the value function and the obstacle. In the case of optimal stopping (only one player) the value function can be characterized as the least excessive (super-harmonic) function. Recently, a similar characterization of the value function was studied for the case of games in [3]. Usually, the probabilistic approach is further used to draw other important conclusions, resembling the analytic approach. More precisely, it can be shown that the value function is a viscosity solution of the HJB. If a comparison results holds, then the value function is *the unique viscosity solution*, and finite-difference numerical methods can be used to approximate it.

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3. Our contribution. Compared to the existing large body of work on optimal stopping (games), we view our contribution as mostly conceptual. We provide here a new approach that lies *in between the analytic and the probabilistic approaches* described above. More precisely, we propose a *probabilistic version of the analytic approach.* We believe our method is novel in that

1. compared to the analytic approach, it does not require the existence of a smooth solution. This is because we do not apply Itô's formula to the solution of the PDE, but *only* to the smooth test functions.

2. compared to the probabilistic approach, we do not perform *any* direct analysis on the value function(s). As a matter of fact, even the very Markov property needed for such analysis is not assumed. The Markov property is hidden behind the uniqueness of the viscosity solution. This is all a consequence of the (same) fact the we apply Itô's lemma to the smooth test functions (as described above) along solutions of SDE without any Markov assumption on the SDE.

We believe our method displays a deeper connection between (stopped) diffusions and (viscosity solutions of) free boundary problems. The fine interplay between how much smoothness is needed for a solution of a PDE in order to apply Itô's formula along the SDE (which is needed in the classical analytic approach) is hidden behind the definition of *stochastic* super- and sub-solutions, which traces back to the seminal work of Stroock and Varadhan [6].

We could summarize the message of our main result as: if a viscosity comparison result for the HJB holds, then there is no need to either find a smooth solution of the HJB, or to analyze the value function(s) to solve the optimization problem. Formally, all classic results hold as expected, i.e., *the unique continuous (but possibly non-smooth) viscosity solution is equal to the value of the game and it is optimal for the player(s) to stop as soon as they reach their corresponding contact/stopping regions.* This amounts to a verification without smoothness, in the spirit of the analytic approach to optimal stopping. This resolution of the problem seems shorter (and more elementary) than the probabilistic approach described above. In addition, our main result tells us that the value function is equal to the infimum over stochastic super-solutions or the supremum over stochastic sub-solutions, resembling the probabilistic results in [3].

Compared to the previous work on Stochastic Perron's method [1], the contribution of the present note lies in *the precise and proper identification of stochastic sub- and super- solutions for the obstacle problem.* The technical contribution consists in proving that, having identified such a definition of stochastic solutions, the Perron's method actually does produce viscosity super- and sub-solutions. The proofs turn out to be very different from [1].

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Pricing of swing options in continuous time

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This talk is devoted to the pricing of swing options in continuous time. In general, the holder of a swing option has the right to exercise a certain total volume up to maturity, but she is subjected to some constraints. Depending on the formulation of the constraints, swing option pricing can be treated as a multiple stopping problem or as a stochastic control problem.

To be more precise, let us assume that the payoff of the option is modeled by an adapted stochastic process $X = (X(t),\ t \in [0,T])$ on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. We suppose that X is nonnegative and the paths of X are rightcontinuous with left limits. Moreover,

$$E[\sup_{0\le t\le T} |X(t)|^2] < \infty.$$

1. Formulation as a multiple stopping problem. Carmona and Touzi [2] suggested to formulate swing option pricing in continuous time via a multiple stopping problem. The holder of the option has the right to exercise the option up to N times. We here use the convention that she receives $X(\tau)/N$, if she exercises the option at time τ. So the aim of the holder of the option is to choose stopping times $\tau_1, \dots, \tau_N$ in an 'admissible' way such that

$$E\left[\frac{1}{N}\sum_{\nu=0}^{N} X(\tau_\nu)\right]$$

is maximized. Here we already assume that the probability measure P is a pricing measure rather than the physical measure, i.e. all tradable and storable assets are σ-martingales under P. It is industry practice to impose a minimal waiting time of $\delta > 0$ in between two exercises. E.g., when exercising a right involves the physical delivery of a commodity, this waiting time, which is known as the refraction period, is usually at least as large as the time required for delivery. Incorporating this constraint leads to the following multiple stopping problem: The price of the swing option with N exercise rights and refraction period δ at a stopping time σ is given by

$$Y^{*,N}(\sigma) := \operatorname*{esssup}_{(\tau_1,\dots,\tau_N)\in\mathcal{S}^N_{\delta,\sigma}} \frac{1}{N}\sum_{\nu=1}^{N} E[X(\tau_\nu)|\mathcal{F}_\sigma], \tag{1}$$

where $\mathcal{S}^N_{\delta,\sigma}$ contains those n-tuples of stopping times $(\tau_1, \dots, \tau_N)$ such that $\tau_1 \ge \sigma$ and $\tau_\nu \ge \tau_{\nu-1} + \delta$ for $\nu = 2, \dots, N$. Here we apply the convention that $X(t) = 0$ for $t > T$, i.e. a right used later than T remains in fact unexercised.

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In this setting we show the following reduction principle to a single stopping problem, which generalizes a result by Carmona and Touzi [2]:

There is an adapted process $X^N(t)$, whose discontinuities from the right are included in the set $D_N := \{T - \nu\delta;\ \nu = 0, \dots, N-1\}$, such that

$$Y^{*,N}(\sigma) = \operatorname*{esssup}_{\tau \in \mathcal{S}_\sigma} \frac{1}{N} E[X^N(\tau)|\mathcal{F}_\sigma] \tag{2}$$

and

$$E[\sup_{0 \le t < \infty} |X^N(t)|^2] < \infty.$$

Here $X^N(t)$ is a modification of $X(t) + (N-1)E[Y^{*,N-1}(t+\delta)|\mathcal{F}_t]$ such that

$$X^N(\tau) = X(\tau) + (N-1)E[Y^{*,N-1}(\tau+\delta)|\mathcal{F}_\tau]$$

for every stopping time τ.

Some technical problems arise in the derivation of this result due to the fact that the Snell envelopes $Y^{*,N}(t)$ may exhibit discontinuities from the right for $N \ge 2$.

We also discuss existence of optimal families of stopping times under the additional assumption that X is leftcontinuous in expectation. Moreover we derive a dual representation for the multiple stopping problem as a minimization problem over martingales and processes of bounded variation, which generalizes a result in discrete time by Schoenmakers [4].

2. Formulation as optimal control problem. Suppose now that the number of exercise rights tends to infinity and the refraction period δ_N tends to zero. If $\lim_{N\to\infty} N\delta_N = \frac{1}{L}$, then the natural limiting problem of (1) is the classical control problem

$$J(\sigma, Y) := \operatorname*{esssup}_{u \in U(\sigma,Y)} E\left[\int_\sigma^T u(s)X(s)ds \middle| \mathcal{F}_\tau\right] \tag{3}$$

where $U(\sigma, Y)$ is the set of all adapted processes with values in $[0, L]$ such that $\int_\sigma^T u(s)ds \le 1 - Y$. Here u can be interpreted as the rate at which the volume is consumed by the holder of the option.

Passing to the limit $N \to \infty$ in (2) suggests that (an appropriate version of) $J(t, y)$ should solve the backward stochastic partial differential equation (BSPDE)

$$\begin{aligned} J(t,y) &= E\left[L\int_t^T (X(s) + D_y^+ J(s,y))_+ ds \middle| \mathcal{F}_t\right], \\ J(T,y) &= 0, \quad J(t,1) = 0. \end{aligned} \tag{4}$$

Here $D_y^+ J$ denotes the right-hand side derivative of J in y. In a Markovian setting this BSPDE reduces formally to a classical Hamilton-Jacobi-Bellman equation. It can also be connected to stochastic Hamilton-Jacobi-Bellman equations in the sense of Peng [3]. We show that the value process $J(t, y)$ solves the BSPDE (4) and that

the right-hand side derivative can be replaced by the left-hand side derivative in (4). To this end we discuss the connection of the discontinuities of $D_y^+ J(s,y)$ in time and space.

We also show that the derivative $D_y J(s,y)$ exists under the additional assumption that X is leftcontinuous in expectation and represent it as an optimal stopping problem of X restricted to some subset of predictable stopping times.

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Control and Nash games with mean field effect

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Mean field theory has raised a lot of interest in the recent years, see in particular Lasry-Lions [11], [12], [13], Gueant-Lasry-Lions [8], Huang-Caines-Malhamé [9], [10], Buckdahn-Li-Peng [7]. There are a lot of applications. In general the applications concern approximating an infinite number of players with common behavior by a representative agent. This agent has to solve a control problem perturbed by a field equation, representing in some way the behavior of the average infinite number of agents. The state equation is modified by the expected value of some functional on the state. We first review in this presentation the linear-quadratic case. This has the advantage of getting explicit solutions. In particular this leads to the study of Riccati equations. We discuss two approaches. One in which the agent considers the mean field term as external, and an equilibrium occurs when this mean field term coincides with the average of his/her own action. The problem reduces to a fixed point. In another one, the mean field is a functional of the state and therefore the agent can influence it by his/her own decision. When there is no control, there is no difference between the two approaches. However, with control the two approaches are not equivalent. In particular, the fixed point approach leads to non-symmetric Riccati equations, which have no control interpretation. They raise interesting mathematical problems of their own. For nonlinear nonquadratic problems, the approach which has been explored is the endogeneous one. The control of the representative agent can influence the mean field term, which is the average of the agents state. The Dynamic pogramming approach fails, because of the so called inconsistency effect. Fortunately, the stochastic maximum principle can be applied. The adjoint variables are solutions of stochastic backward differential equations, with mean field terms. In the approach of Lasry-Lions, the starting point is a Nash equilibrium game for a very large number of players. In principle, the problem can be treated by Dynamic Programming. The Bellman equation becomes a system of nonlinear partial differential equations, for which the techniques of [2] can be considered. When the number of players becomes infinite, and all of them are identical, then going to the limit, one obtains an Hamilton-Jacobi-Bellman equation, with mean field term. The mean field term is reminiscent of the coupling with other players, which existed before going to the limit. We compare the various approaches, and their interprations as control problems. In Lasry-Lions approach the limit is obtained thanks to ergodic theory, which means that the limit control problem is an ergodic control problem, with mean field effect.

There is a different and interesting approach which also leads to similar types of P.D.E with mean-field terms. The state equation is the Chapman-Kolmogorov

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equation, describing the probability measure of the state. It is the dual control problem. Then, the Bellman equation can be interpreted as a necessary condition of optimality for the dual problem. To generate mean-field terms, it is sufficient to consider objective functions which are not just linear in the probability measure, but more complex.

This approach has a different type of application. In the traditional stochastic control problem, the objective functional is the expected value of a cost depending on the trajectory. So it is linear in the probability measure. This type of functional leaves out many current considerations in control theory, namely situations where one wants to take into consideration not just the expected value, but also the variance. This case occurs often in Risk Management. Moreover, one may be interested by several functionals on the trajectory, even though one is satisfied with expected values. If one combines these various expected values in a single pay-off, one is lead naturally to mean-field problems. They are meaningful even without considering ergodic theory, i. e. long term behavior.

Anyway, in all the previous considerations, the averaging approach reduces an infinite number agent to a representative agent, who has a control problem to solve, with an external effect, representing the averaged impact of the infinite number of players. Of course, this framework relies on the assumption that the players behave in a similar way. By construction, it eliminates the situation of a remaining Nash equilibrium for a finite number of players, with mean field terms.

In most real problems of economics, there is not just one representative agent and a large community of identical players, which impact with a mean field term. There is the situation of several major players, and large communities.

So a natural question is to consider the problem of these major players. They know that they can influence the community, and they also compete with each other. So the issue is that of differential games, with mean field terms, and not of mean field equations arising from the limit of a Nash equilibrium for an infinite number of players.

One way to recover this system of nonlinear P.D.E. with mean field terms is to consider averaging 2 within groups. Each of them is composed of an homogeneous community, but different communities are not identical.

To recover the system of nonlinear P.D.E. it is easier to proceed with the dual problems as explained above. One can consider a differential game for state equations which are probability distributions of states, and evolve according to Chapman-Kolmogorov equations. One recovers nonlinear systems of P.D.E. with mean field terms, with a different motivation. Another interesting feature of this approach is that we do not need to consider an ergodic situation, as it is the case in the standard approach of mean field theory. In fact, considering strictly positive discounts is quite meaningful in our applications. This leads to systems of non-linear P.D.E. with mean field coupling terms, that we can study with a minimum set of assumptions. The ergodic case, when the discount vanishes, requires much stringent assumptions, as it is already the case when there is no mean field term. We refer to Bensoussan-Frehse [2], [4] and Bensoussan-Frehse-Vogelgesang [5], [6]

for the situation without mean field term. Basically our set of assumptions remains valid and we have to incorporate additional assumptions to deal with the mean field terms.

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Liquidity, equilibrium and asymmetric information

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1. To formulate the model of the market precisely, let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,1]}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness. Assume that on this probability space there exist a continuous random variable $V \in \mathcal{F}_0$ and a standard Brownian motion B, independent of V.

We consider a market in which a single risky asset is traded. The value of this asset, V, will be a public knowledge at some future time $t = 1$. For simplicity of exposition we assume that the risk free interest rate is 0.

There are three types of agents that interact in this market:

i) Liquidity traders, whose demands are random, price inelastic and do not reveal any information about the value of V. In particular we assume that their cumulative demand at time t is given by $Z_t = \sigma B_t$.

ii) A single insider who knows V at time $t = 0$ and is risk neutral. We will denote insider's cumulative demand at time t by X_t. The filtration of the insider, $\mathcal{F}^I$, is generated by observing the price of the risky assset and V.

iii) Market makers observe the net supply of the risky asset, $Y = X + Z$, thus, their filtration, $\mathcal{F}^M$, is generated by Y.
We also assume that the market makers have identical CARA utilities with the common risk aversion parameter ρ, and compete in a Bertrand fashion for the net supply of the risky asset. In case of several market makers quoting the same winning price, we adopt the convention that the total order is equally split among them. As a result of this competition in the equilibrium each market maker quotes the price which achieves zero utility gain and, therefore, Y is split equally. The number of market makers is assumed to be $N \geq 2$.

2. The assumption that the markets makers observe only the net supply implies that they cannot separate the informed and uninformed trades. Hence, their quotes at time t can only depend on $(Y_s)_{s=0}^t$. However, we would be looking at only Markovian equilibrium, thus, we consider only the quotes of the form $H(t, Y_t)$. Additionally, we assume that H is smooth enough, i.e. $H(t, y) \in C^{1,2}$, it is strictly increasing in y, and satisfies

$$\mathbb{E}H^2(1, Z_1) < \infty \qquad \text{and} \qquad \mathbb{E}\int_0^1 H^2(t, Z_t)\, dt < \infty. \tag{1}$$

The class of such functions is denoted with $\mathcal{H}$ and any $H \in \mathcal{H}$ is called a *pricing rule*.

As any $H \in \mathcal{H}$ is invertible, observing price is equivalent to observing Y and therefore the insider can perfectly infer the demand of the liquidity traders since she

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knows her own demand. It follows from this consideration that $\mathcal{F}_t^I = \sigma(V, Z_s; s \leq t)$. Obviously, for an insider strategy, X, to be admissible, it has to be adapted to $\mathcal{F}^I$. We further require that for a given $H \in \mathcal{H}$

$$\mathbb{E}^v \int_0^1 H^2(t, X_t + Z_t)\, dt < \infty, \tag{2}$$

where $\mathbb{E}^v$ is the expectation taken with respect to $\mathbb{P}^v$ associated to the insider given the realisation $V = v$. Moreover, X is absolutely continuous , i.e. $X_t = \int_0^t \alpha_s\, ds$ (This restriction to the set of absolutely continuous strategies is without loss of generality since strategies with a martingale component and/or jumps are strictly suboptimal as shown in [1]). For any given $H \in \mathcal{H}$ a strategy satisfying the above conditions is called admissible and the class of admissible strategies will be denoted by $\mathcal{A}(H)$. Observe that if $X \in \mathcal{A}(H)$ then the terminal wealth of the insider is given by

$$W_1^X := \int_0^1 X_s dH(s, Y_s) + X_1(V - H(1, X_1)) = \int_0^1 (V - H(s, X_s))\, dX_s. \tag{3}$$

3. By an equilibrium we mean a pair (H^*, X^*) for $H^* \in \mathcal{H}$ and $X^* \in \mathcal{A}(H^*)$ such that

i) given H^*, the insider's strategy X^* solves her optimization problem:

$$\mathbb{E}^v[W_1^{X^*}] = \sup_{X \in \mathcal{A}(H^*)} \mathbb{E}^v[W_1^X].$$

ii) Given X^*, the pricing rule H^* satisfies zero-utility gain condition, i.e. $(U(G_t))_{t=0}^1$ is a $(\mathcal{F}^M, \mathbb{P})$-martingale, where

$$G_t := -\frac{1}{N} \int_0^t Y_s^*\, dH^*(s, Y_s^*) + \mathbf{1}_{t=1} \frac{Y_1^*}{N} (H^*(Y_1^*, 1) - V).$$

It is shown that in equilibrium the insider drives the total demand so that $H(1, Y_1) = V$, i.e. the market price converges to the true price. Given this observation, the existence of equilibrium will follow from the following theorem:

Theorem 1. *Suppose that $V = f(\eta)$, where η is a standard normal random variable and f is a strictly increasing function which is either linear or bounded with a continuous derivative. Then, there is a pair (H, Y) which solves the following system:*

$$H_t + \frac{1}{2}\sigma^2 H_{yy} = 0 \tag{4}$$

$$d\xi_t = \sigma dB_t - \frac{\sigma^2 \rho}{2N} \xi_t H_y(t, \xi_t) dt \tag{5}$$

$$V \stackrel{d}{=} H(1, \xi_1), \tag{6}$$

where $\stackrel{d}{=}$ *stands for equality in distribution.*

Moreover, H_y *is bounded with* $0 < H_y(t,y)$ *for all* $(t,y) \in [0,1] \times \mathbb{R}$ *and* ξ *is the unique strong solution of (5). Furthermore,* ξ *admits a transition density* $p(s,y;t,z)$ *such that, for any fixed* (t,z), $p(s,y;t,z) > 0$ *on* $[0,t) \times \mathbb{R}$ *and is* $C^{1,2}([0,t) \times \mathbb{R})$.

The optimal demand of the insider and the pricing rule in equilibrium are given in the following theorem.

Theorem 2. *Suppose that* $V = f(\eta)$, *where* η *is a standard normal random variable and* f *is either linear or a bounded function with a continuous derivative. Then,* (H^*, X^*) *is an equilibrium where*

$$X_t^* = \int_0^t \left\{ -\frac{\sigma^2 \rho}{2N} Y_s^* H_y^*(s, Y_s^*) + \sigma^2 \frac{p_y}{p}(s, Y_s^*; 1, H^{*-1}(1, V)) \right\} ds$$

and H^* *and* p *are the functions defined in Theorem 1.*

Moreover, under $\mathcal{F}^M$ *the equilibrium demand evolves as*

$$Y_t^* = \sigma B_t^Y - \frac{\sigma^2 \rho}{2N} \int_0^t Y_s^* H_y^*(s, Y_s^*)\, ds.$$

In particular, when $f(y) = ay + b$, *then* $H^*(t,y) = \lambda y + b$, *where* λ *is the unique solution to*

$$1 - e^{-\frac{\rho\sigma^2}{N}\lambda} = \frac{\rho a^2}{N} \frac{1}{\lambda}, \tag{7}$$

and the equilibrium demand Y^* *solves*

$$dY_t^* = \sigma dB_t + \frac{\rho\sigma^2}{2N} \frac{a\eta - \lambda Y_t^* \cosh\left(\frac{\rho\sigma^2\lambda}{2N}(1-t)\right)}{\sinh\left(\frac{\rho\sigma^2\lambda}{2N}(1-t)\right)}. \tag{8}$$

4. In this talk I will also discuss the effect of risk aversion of market makers on various liquidity parameters such as depth and resilience. The comparison with the corresponding features of the model considered in [1] will also be given.

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Stochastic optimization of sailing trajectories in an upwind regatta

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1. In a sailboat regatta, the navigator attempts to plan out the fastest possible course, based on a forecast of wind and weather conditions, as well as to provide real-time updates when these conditions change. It turns out that some wind changes are typically as unpredictable as stock market movements, so it is natural to develop probabilistic models for wind and weather in order to provide tools to help the navigator make decisions.

The velocity of a sailboat that wants to follow a specific bearing is directly influenced by the wind speed and direction. Since a sailboat cannot advance directly into the wind, a boat sailing in an upwind leg must sail at an angle to the wind (of about 30° degrees) and follow a zig-zag trajectory consisting of a sequence of "tackings," that is, of changes in direction such that the bow of the yacht crosses through the eye of the wind. Each tack implies a loss of speed, hence of time, and the decisions concerning when to tack are a crucial part of the navigator's recommendations. Further, since wind direction and speed fluctuate over time, the choice of bearing is difficult to determine intuitively and must be modified during the race.

In this talk, we report on a project to formulate this trajectory optimization problem in the framework of stochastic control theory. The objective was twofold: first, to develop statistical models of wind behavior and use them to perform stochastic optimization under real-world conditions, so as to develop an onboard decision tool that could provide the navigator of the Swiss team Alinghi with real-time recommendations for an America's Cup race, and second, to identify simpler mathematical models of wind behavior that are amenable to a complete mathematical analysis by stochastic optimization methods, with the identification of optimal strategies and rigorous proofs that these strategies are best possible within the model.

2. A statistical analysis of wind behavior was carried out, in collaboration with S. Morgenthaler and S. Sardy at the Ecole Polytechnique Fédérale de Lausanne. Because races occur in the months of May to July, and weather conditions are different during the rest of the year, relevant data can only be collected during these months. Furthermore, in the years that preceded the race, few weather stations were operational and so past data was available only for a limited number of days. Races occur during the afternoon, so a model for wind that would be accurate during a two-hour afternoon race period was needed.

On a race day, the morning's wind could be used to help predict the wind behavior during the afternoon. On the other hand, racing teams were allowed to

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communicate with the outside world only up to five minutes before the race, and after that, could only rely on onboard instruments.

Various continuous-time models have been compared, and turn out to depend on the geographic location: the model we used was different than that suggested in [2]. In the end, for numerical purposes, the evolution of the wind speed and wind direction over time were approximated by discrete-time Markov chains. The transition matrices of these chains were chosen among a small number of possibilities, corresponding to days classified according to low, medium or high volatility of the wind. The data collected on the morning before the race was used to decide the level of volatility for that day. A further effort was made in the last minutes before the break in communication to decide if the wind direction exhibited a trend to the left or right, and the amplitude of that trend.

3. The statistical study produced, for any given day, stochastic processes representing the evolution of wind direction and wind speed over time. In order to carry out numerical computations, the model had to be discretized. Since the position of the boat is essentially a point in the plane, a discretization of the race area was also needed. This discretization had to be compatible with key features of sailboat motion, and fine enough to capture the essential behaviors. In addition, the navigator recommends the bearing that the boat is to follow, so the action space is also continuous, but the model only allowed for a small number of relevant choices. Since the wind direction is measured with a precision of a few degrees, the wind direction was modeled by a discrete-time Markov chain for which each step represented a transition in wind direction and speed over a thirty second time-interval.

4. With the model in hand, offline computations could determine the optimal action for every possible position of the boat on the race field, and every possible wind direction and wind speed. This provided a database that could be used onboard the boat in real time. The onboard computer system could feed in real time wind data, and the model would provide the navigator with a recommendation on the optimal action to be taken. It would also quantify the advantage of using this action relative to other actions, indicate how far ahead one boat is relative to the other, and give advance notice of course changes expected in the near future. These results are reported in [1].

5. In order to carry out a rigorous mathematical analysis, simpler models of wind behavior are useful. This was the objective of [3]. The simplest model for the evolution of wind angle is a two-state continuous time Markov chain. This already leads to a complex free-boundary problem, in which the value function can be written as the solution to a system of hyperbolic partial differential equations with free boundaries, from which many features of the optimal strategy can be obtained. A second natural model is when the wind direction evolves as a Brownian motion on a circle. In this case, the value function solves a system of parabolic partial differential equations with free boundaries. In both cases, the principle of smooth fit can be used to help determine the free boundaries and properties of the value function. This research is reported in [3].

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Average-cost Markov decision processes with weakly continuous transition probabilities

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This talk presents general sufficient conditions for the existence of stationary optimal policies for discounted and average-reward Markov Decision Process with Borel state sets and with weakly continuous transition probabilities. The results for average costs per unit time extend Scäl's [10] sufficient conditions for the existence of stationary optimal policies to problems with noncompact actions sets. For setwise continuous transition probabilities, similar results were established in Scäl [10] for compact action sets and extended in Hernández-Lerma [5] to general action sets. This talk is based on Feinber, Kasyanov and Zadoianchuk [2].

Consider a *discrete-time MDP* $(\mathbb{X}, \mathbb{Y}, \Phi, q, u)$ with a *state space* $\mathbb{X}$, an *action space* $\mathbb{Y}$, one-step costs u, and transition pobabilities q. The terminology is the same as in [2, 4] and references therein. Assume that $\mathbb{X}$ and $\mathbb{Y}$ are *Borel subsets* of Polish (complete separable metric) spaces. For a topological space $\mathbb{U}$ we denote by $\mathcal{B}(\mathbb{U})$ its Borel σ-field. For all $x \in \mathbb{X}$ a nonempty Borel subset $\Phi(x)$ of $\mathbb{Y}$ represents the *set of actions* available at x. Assume also that $\mathrm{Gr}_{\mathbb{X}}(\Phi) = \{(x, y) : x \in \mathbb{X}, y \in \Phi(x)\}$ is a measurable subset of $\mathbb{X} \times \mathbb{Y}$, that is, $\mathrm{Gr}_{\mathbb{X}}(\Phi) \in \mathcal{B}(\mathbb{X} \times \mathbb{Y})$, where $\mathcal{B}(\mathbb{X} \times \mathbb{Y}) = \mathcal{B}(\mathbb{X}) \otimes \mathcal{B}(\mathbb{Y})$; and there exists a measurable mapping $\phi : \mathbb{X} \to \mathbb{Y}$ such that $\phi(x) \in \Phi(x)$ for all $x \in \mathbb{X}$. The *one step cost*, $u(x, y) \leq +\infty$, for choosing an action $y \in \Phi(x)$ in a state $x \in \mathbb{X}$, is a *bounded below measurable* function on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$. Let $q(B|x, y)$ be the *transition kernel* representing the probability that the next state is in $B \in \mathcal{B}(\mathbb{X})$, given that the action y is chosen in the state x. This means that $q(\cdot|x, y)$ is a probability measure on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ for all $(x, y) \in \mathrm{Gr}_{\mathbb{X}}(\Phi)$; and $q(B|\cdot, \cdot)$ is a Borel function on $(\mathrm{Gr}_{\mathbb{X}}(\Phi), \mathcal{B}(\mathrm{Gr}_{\mathbb{X}}(\Phi)))$ for all $B \in \mathcal{B}(\mathbb{X})$.

Let $\mathrm{Gr}_Z(\Phi) = \{(x, y) \in Z \times \mathbb{Y} : y \in \Phi(x)\}$, where $Z \subseteq \mathbb{X}$. For a topological space $\mathbb{U}$, we denote by $\mathbb{K}(\mathbb{U})$ *the family of all nonempty compact subsets of* $\mathbb{U}$.

For an $\overline{\mathbb{R}}$-valued function f, defined on a nonempty subset U of a topological space $\mathbb{U}$, consider the level sets $\mathcal{D}_f(\lambda; U) = \{y \in U : f(y) \leq \lambda\}$, $\lambda \in \mathbb{R}$. We recall that a function f is *lower semi-continuous (l.s.c.) on* U if all the level sets $\mathcal{D}_f(\lambda; U)$ are closed, and a function f is *inf-compact on* U (lower semi-compact cf. [12]) if all these sets are compact.

Definition 1. *A function* $u : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ *is called* $\mathbb{K}$*-inf-compact on* $\mathrm{Gr}_{\mathbb{X}}(\Phi)$*, if for every* $K \in \mathbb{K}(\mathbb{X})$ *this function is inf-compact on* $\mathrm{Gr}_K(\Phi)$.

We set $\Phi^{\#}(x) = \{y \in \Phi(x) : v(x) = u(x, y)\}$. The first statement of the following theorem extends the well-known Berge's theorem of the minimum [1, Theorem 2, p. 116] or [7, Proposition 3.3, p. 83] to noncompact image (or decision) sets. The proofs and additional details can be found in [3].

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Theorem 1. *If the function $u : \mathbb{X} \times \mathbb{Y} \to \overline{\mathbb{R}}$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$, then the function $v : \mathbb{X} \to \overline{\mathbb{R}}$ is l.s.c. If moreover u is a continuous function on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$ and $\Phi : \mathbb{X} \to 2^{\mathbb{Y}} \setminus \emptyset$ is l.s.c., then the function v is continuous on $\mathbb{X}$ and the solution multifunction $\Phi^{\#} : \mathbb{X} \to \mathbb{K}(\mathbb{Y})$ has a closed graph. Additionally, if Φ is upper semi-continuous (u.s.c.), then $\Phi^{\#}$ is u.s.c.*

The following lemma provides a useful criterium for $\mathbb{K}$-inf-compactness of u on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$, when the spaces $\mathbb{X}$ and $\mathbb{Y}$ are metrizable. In this form the $\mathbb{K}$-inf-compactness assumption is introduced in Feinberg, Kasyanov and Zadoianchuk [2] as Assumption $(\mathbf{W}^*)$(ii).

Lemma 1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be metrizable spaces. Then u is $\mathbb{K}$-inf-compact on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$ if and only if the following two conditions hold: (i) u is l.s.c. on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$; (ii) if a sequence $\{x_n\}_{n\geq 1}$ with values in $\mathbb{X}$ converges and its limit x belongs to $\mathbb{X}$ then any sequence $\{y_n\}_{n\geq 1}$ with $y_n \in \Phi(x_n)$, $n \geq 1$, satisfying the condition that the sequence $\{u(x_n, y_n)\}_{n\geq 1}$ is bounded above, has a limit point $y \in \Phi(x)$.*

We also suppose the following assumption that implies the existence of stationary optimal policies for discounted MDPs.

Assumption $(\mathbf{W}^*)$. *(i) u is bounded below and $\mathbb{K}$-inf-compact on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$; (ii) the transition probability $q(\cdot|x, y)$ is weakly continuous in $(x, y) \in \mathrm{Gr}_{\mathbb{X}}(\Phi)$.*

Weak continuity of q in (x, y) means that $\int_{\mathbb{X}} f(z)q(dz|x_k, y_k) \to \int_{\mathbb{X}} f(z)q(dz|x, y)$, $k \to +\infty$, for any sequence $\{(x_k, y_k), k \geq 1\}$ converging to (x, y), where (x_k, y_k), $(x, y) \in \mathrm{Gr}_{\mathbb{X}}(\Phi)$, and for any bounded continuous function $f : \mathbb{X} \to \mathbb{R}$.

Denote the class of all l.s.c. and bounded below functions $\varphi : \mathbb{X} \to \overline{\mathbb{R}}$ with $\mathrm{dom}\,\varphi := \{x \in \mathbb{X} : \varphi(x) < +\infty\} \neq \emptyset$ by $L(\mathbb{X})$. Let $\mathbb{F}$ be a family of Borel mappings $\phi : \mathbb{X} \to \mathbb{Y}$ such that $\phi(x) \in \Phi(x)$ for all $x \in \mathbb{X}$.

An important consequence of Assumption $(\mathbf{W}^*)$ is that it implies that $\mathbb{F}$ contains suitable "minimizers". The following lemma is useful for establishing continuity properties of the value functions; for later relevant results see Feinberg et al. [2]. The proof of this lemma follows from Theorem 1 and from the Arsenin-Kunugui theorem (Kechris [8, p. 297]).

Lemma 2. *If Assumption $(\mathbf{W}^*)$ holds and $\underline{u} \in L(\mathbb{X})$, then the function $(x, y) \to u(x, y) + \int_{\mathbb{X}} \underline{u}(z)q(dz|x, y)$ is $\mathbb{K}$-inf-compact on $\mathrm{Gr}_{\mathbb{X}}(\Phi)$ and the nonempty sets*

$$\Phi_*(x) = \left\{ y \in \Phi(x) : \underline{u}^*(x) = u(x, y) + \int_{\mathbb{X}} \underline{u}(z)q(dz|x, y) \right\}, \qquad x \in \mathbb{X}, \tag{1}$$

satisfy the following properties: (a) $\mathrm{Gr}_{\mathbb{X}}(\Phi_*)$ is a Borel subset of $\mathbb{X} \times \mathbb{Y}$; (b) if $\underline{u}^*(x) = +\infty$, then $\Phi_*(x) = \Phi(x)$, and, if $\underline{u}^*(x) < +\infty$, then $\Phi_*(x)$ is compact.

As usual a *policy* is a sequence $\pi = \{\pi_n\}_{n=0,1,\ldots}$ of decision rules (cf. [2, 4] and references therein), where for each $n = 0, 1, \ldots$ $\pi_n(\cdot|h_n)$ is a conditional probability on $(\mathbb{Y}; \mathcal{B}(\mathbb{Y}))$, given the history $h_n = (x_0, y_0, x_1, y_1, \ldots, y_{n-1}, x_n)$, satisfying $\pi_n(\Phi(x_n)|h_n) = 1$. The class of *all policies* is denoted by Π. Moreover, π is called *nonrandomized*, if each probability measure $\pi_n(\cdot|h_n)$ is concentrated at one point. A nonrandomized policy is called *Markov*, if all of the decisions depend on the current state and time only. A Markov policy is called *stationary*, if all the decisions depend on the current state only. Thus, a Markov policy π is defined by a sequence

$\phi_0, \phi_1, \ldots$ of Borel mappings $\phi_n \in \mathbb{F}$. A stationary policy π is defined by a Borel mapping $\phi \in \mathbb{F}$.

For a policy π, given initial state $x_0 = x \in \mathbb{X}$, for a finite horizon $N \geq 0$ let us define the *expected total discounted costs* $v^{\pi}_{N,\alpha} := \mathbb{E}^{\pi}_{x} \sum\limits_{n=0}^{N-1} \alpha^n u(x_n, y_n)$, $x \in \mathbb{X}$, where $\alpha \geq 0$ is the discount factor and $v^{\pi}_{0,\alpha}(x) = 0$. When $N = \infty$ and $\alpha \in [0,1)$, $v^{\pi}_{N,\alpha}$ defines an *infinite horizon expected total discounted cost* denoted by $v^{\pi}_{\alpha}(x)$. The *average cost per unit time* is defined as $w^{\pi}(x) := \limsup\limits_{N\to+\infty} \frac{1}{N} v^{\pi}_{N,1}(x)$, $x \in \mathbb{X}$. For any function $\triangle^{\pi}(x)$, including $\triangle^{\pi}(x) = v^{\pi}_{N,\alpha}(x)$, $\triangle^{\pi}(x) = v^{\pi}_{\alpha}(x)$, and $\triangle^{\pi}(x) = w^{\pi}(x)$, define the *optimal cost* $\triangle(x) := \inf\limits_{\pi\in\Pi} \triangle^{\pi}(x)$, $x \in \mathbb{X}$. A policy π is called *optimal* for the respective criterion, if $\triangle^{\pi}(x) = \triangle(x)$ for all $x \in \mathbb{X}$. For $\triangle^{\pi} = v^{\pi}_{n,\alpha}$, the optimal policy is called *n-horizon discount-optimal*; for $\triangle^{\pi} = v^{\pi}_{\alpha}$, it is called *discount-optimal*; for $\triangle^{\pi} = w^{\pi}$, it is called *average-cost optimal* [2, 4, 5, 6, 10]. These definitions of optimality are standard.

Assumption ($\underline{\mathbf{B}}$). (a) $w^* := \inf\limits_{x\in\mathbb{X}} w(x) < \infty$, (b) $\liminf\limits_{\alpha\uparrow 1} u_{\alpha}(x) < \infty$ $\forall x \in \mathbb{X}$.

Assumption ($\underline{\mathbf{B}}$)(a) is equivalent to the existence of $x \in \mathbb{X}$ and $\pi \in \Pi$ with $w^{\pi}(x) < \infty$. If Assumption ($\underline{\mathbf{B}}$)(a) does not hold then the problem is trivial, because $w(x) = \infty$ for all $x \in \mathbb{X}$ and any policy π is average-cost optimal.

To state the main result we also need the following notation [10]: for $\alpha \in [0,1)$: $m_{\alpha} = \inf\limits_{x\in\mathbb{X}} v_{\alpha}(x)$, $u_{\alpha}(x) = v_{\alpha}(x) - m_{\alpha}$, $\underline{w} = \liminf\limits_{\alpha\uparrow 1}(1-\alpha)m_{\alpha}$, $\overline{w} = \limsup\limits_{\alpha\uparrow 1}(1-\alpha)m_{\alpha}$. Observe that $u_{\alpha}(x) \geq 0$ for all $x \in \mathbb{X}$. Schäl [10, Lemma 1.2] and Assumption ($\underline{\mathbf{B}}$)(a) implies $0 \leq \underline{w} \leq \overline{w} \leq w^* < +\infty$. According to Schäl [10, Proposition 1.3], under Assumption ($\underline{\mathbf{B}}$)(a), if there exists a measurable function $g : \mathbb{X} \to [0, +\infty)$ and a stationary policy ϕ such that $\underline{w} + g(x) \geq u(x, \phi(x)) + \int_{\mathbb{X}} g(z) q(dz|x, \phi(x))$, $x \in \mathbb{X}$, then ϕ is *average-cost optimal* and $w(x) = w^* = \underline{w} = \overline{w}$ for all $x \in \mathbb{X}$. Here we need a different form of such a statement.

Theorem 2. *Let Assumption ($\underline{\mathbf{B}}$)(a) holds. If there exists a measurable function $g : \mathbb{X} \to [0, +\infty)$ and a stationary policy ϕ such that*

$$\overline{w} + g(x) \geq u(x, \phi(x)) + \int_{\mathbb{X}} g(z) q(dz|x, \phi(x)), \quad x \in \mathbb{X}, \tag{2}$$

then ϕ is average-cost optimal and

$$w(x) = w^{\phi}(x) = \limsup_{\alpha\uparrow 1}(1-\alpha) v_{\alpha}(x) = \overline{w} = w^*, \quad x \in \mathbb{X}. \tag{3}$$

Assumption ($\mathbf{W}^*$) and "boundedness" Assumption ($\underline{\mathbf{B}}$) on the function u_{α}, which is weaker than the boundedness Assumption ($\mathbf{B}$) introduced by Schäl [10], lead to the validity of stationary average-cost optimal inequalities and the existence of stationary policies.

Let us set $\Phi^*(x) := \left\{ y \in \Phi(x) \,:\, \overline{w} + \underline{u}(x) \geq u(x,y) + \int_{\mathbb{X}} \underline{u}(z) q(dz|x,y) \right\}$, $\underline{u}(x) := \liminf\limits_{\alpha\uparrow 1,\ z\to x} u_{\alpha}(z)$, $x \in \mathbb{X}$, and let $\Phi_*(x)$, $x \in \mathbb{X}$, be the sets defined in (1) for this function $\underline{u}$; $\Phi_*(x) \subseteq \Phi^*(x)$.

Theorem 3. *Suppose Assumptions* ($\mathbf{W}^*$) *and* ($\underline{\mathbf{B}}$) *hold. There exist a stationary policy* ϕ *satisfying (2). Thus, equalities (3) hold for this policy* ϕ*. Furthermore, the following statements hold:* (a) *the function* $\underline{u} : \mathbb{X} \to \mathbb{R}_+$ *is l.s.c.;* (b) *the nonempty sets* $\Phi^*(x)$*,* $x \in \mathbb{X}$*, satisfy the following properties:* (b_1) *the graph* $\mathrm{Gr}_{\mathbb{X}}(\Phi^*)$ *is a Borel subset of* $\mathbb{X}\times\mathbb{Y}$*;* ($\mathrm{b}_2$) *for each* $x \in \mathbb{X}$ *the set* $\Phi^*(x)$ *is compact;* (c) *a stationary policy* ϕ *is optimal for average costs and satisfies (2), if* $\phi(x) \in \Phi^*(x)$ *for all* $x \in \mathbb{X}$*;* (d) *there exists a stationary policy* ϕ *with* $\phi(x) \in \Phi_*(x) \subseteq \Phi^*(x)$ *for all* $x \in \mathbb{X}$*;* (e) *if, in addition,* u *is inf-compact on* $\mathrm{Gr}_{\mathbb{X}}(\Phi)$*, then the function* $\underline{u}$ *is inf-compact.*

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Equilibrium stochastic behaviors in repeated games

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1. Theory of repeated games essentially motivated by studies on biological adaptation and economic behavior concentrates on analyses of long-term dynamics in societies of repeatedly interacting agents (players), which follow their individual boundedly rational behavior strategies (see, e.g., [1] – [6]). If the players are allowed to choose their boundedly rational behavior strategies within given sets, their mutually acceptable choices are naturally associated with the behavior strategies forming a game-theoretic equilibrium with respect to the players' individual long-term performance criteria. Long-term equilibria in repeated games with non-specified deterministic behavior strategies were defined in [7].

In this paper, we focus on infinite repeated bimatrics games with non-specified stochastic behavior starategies, in which the expectations of the players' benefits averaged over the game rounds (the players' expected avaraged benefits) serve as the players' long-term performance criteria.

First, we provide conditions sufficient for a (Nash) equilibrium pair of the players' behavior strategies to exist within given sets of the players' admissible behavior strategies; the conditions require in particular that all the players' admissible behavior strategies are strictly randomized, and the sets of the players' admissible behavior strategies satisfy an appropriate convexity assumption.

Next, we consider a particular infinite repeated 2×2-bimatrix game with non-specified behaviour strategies. We depart from coupling the players' particular, 'traditional' deterministic boundedly rational behavior strategies – the 'best reply' ones. Then we allow the players to choose their stochastic behavior strategies in 'neigborhoods' of the 'traditional' ones and characterize the equilibrium behavior strategies. In particular, we find that the equilibrium behavior strategies differ from the 'traditional' deterministic ones and have necessarily non-trivial stochastic components.

The paper incorporates results of [8] – [10].

2. We start off with a bimatrix game given by benefit matrices $A = (a_{ij})_{i \in X_1, j \in X_2}$ and $B = (b_{ij})_{i \in X_1, j \in X_2}$ where $X_1 = \{1, \dots, n\}$, $X_2 = \{1, \dots, m\}$ with some natural n and m; here the row index, $i \in X_1$, and column index, $j \in X_2$, stand for pure strategies of player 1 and player 2, respectively; and a_{ij} and b_{ij} denote, respectively, the benefits player 1 and player 2 receive provided they use their pure strategies i and j, respectively. As usual, we understand mixed strategies of players 1 and 2 as probability measures on the sets of their pure strategies, X_1 and X_2, respectively.

We define *behavior strategies* of player 1 and player 2 as arbitrary maps of $X_1 \times X_2$ into the sets of all mixed strategies of players 1 and 2, respectively. Given

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an *initial* pair of the players' pure strategies, $(i_0, j_0) \in X_1 \times X_2$, every behavior strategy of player 1, $p : (i,j) \mapsto p_{ij}$, and every behavior strategy of player 2, $q : (i,j) \mapsto q_{ij}$, determine an *infinite repeated game*; the latter is defined as the infinite discrete-time Markov process on $X_1 \times X_2$, with times (also called *rounds*) $0, 1, \ldots$, the initial state (i_0, j_0) and the transition probability $(i,j) \mapsto p_{ij} \times q_{ij}$.

In the infinite repeated game corresponding to the players' behavior strategies p and q, the *expected average benefits* of players 1 and 2 on round $k \geq 1$ are, respectively, the expectations of the random variables $((i_0, j_0), (i_1, j_1), \ldots) \mapsto (a_{i_1 j_1} + \ldots + a_{i_k j_k})/k$ and $((i_0, j_0), (i_1, j_1), \ldots) \mapsto (b_{i_1 j_1} + \ldots + b_{i_k j_k})/k$ on the probability space of the repeated game's trajectories $((i_0, j_0), (i_1, j_1), \ldots)$. Using properties of finite-state Markov processes (see [1]), one can show that the expected average benefits of players 1 and 2 on round k converge to some limits, which we denote, respectively, $J_1(p,q)$ and $J_2(p,q)$, as $k \to \infty$; $J_1(p,q)$ and $J_2(p,q)$ represent the *expected average benefits* of, respectively, players 1 and 2 in the infinite repeated game corresponding to the players' behavior strategies p and q.

Let players 1 and 2 be allowed to choose their behavior strategies within given sets of *admissible behavior strategies*, S_1 and S_2, respectively. In this situation each player is interested in choosing his/her admissible behavior strategy so as to maximize his/her expected average benefit. A game-theoretic interpretation of that is a *behavior game*, in which the actions (strategies) of players 1 and 2, p and q, vary within S_1 and S_2, respectively, and the benefit functions for players 1 and 2 are $(p,q) \mapsto J_1(p,q)$ and $(p,q) \mapsto J_2(p,q)$, respectively.

3. Consider the issue of the existence of a Nash equilibrium in the behavior game. Following a standard game-theoretical definition, we call a pair $(p^*, q^*) \in S_1 \times S_2$ to be a *Nash equilibrium* (in the behavior game) if p^* maximizes $p \mapsto J_1(p, q^*)$ on S_1 and q^* maximizes $q \mapsto J_2(p^*, q)$ on S_2.

Let us give several definitions. We shall say that S_1 (respectively, S_2) is *strictly randomized* if every $p \in S_1$ (respectively, every $q \in S_2$) takes values in the set of all strictly mixed strategies of player 1 (respectively, player 2).

We shall say that S_1 (respectively, S_2) is *parallepipedally convex* if for every $p^{(1)}, p^{(2)} \in P$ (respectively, $q^{(1)}, q^{(2)} \in Q$) and every family $(\lambda_{ij})_{(i,j) \in X_1 \times X_2}$ in $[0,1]$ the map $(i,j) \mapsto \lambda_{ij} p_{ij}^{(1)} + (1 - \lambda_{ij}) p_{ij}^{(2)}$ (respectively, $(i,j) \mapsto \lambda_{ij} q_{ij}^{(1)} + (1 - \lambda_{ij}) q_{ij}^{(2)}$) lies in P (respectively, in Q).

We shall say that S_1 (respectively, S_2) is *closed* if $\Pi_{(i,j) \in X_1 \times X_2} \{p_{ij} : p \in P\}$ is closed in $(R^n)^{nm}$ (respectively, $\Pi_{(i,j) \in X_1 \times X_2} \{q_{ij} : q \in Q\}$) is closed in $(R^m)^{nm}$).

Our existence theorem reads as follows.

Theorem 1. *Let S_1 and S_2 be strictly randomized, parallelepipedally convex and closed. Then the behavior game has a Nash equilibrium.*

4. Consider a situation where players 1 and 2 dominated by historically justified 'traditional' behavior paradigms explore if small 'innovations' in their 'traditional' behaviors can improve their performance in the long run.

Let each player have two pure strategies in the original bimatrix game, i.e., $X_1 =$

$X_2 = \{1, 2\}$, and the latter bimatrix game have the single mixed equilibrium; then, with no loss in generality (see [12]), we set $a_{11} > a_{21}$, $a_{22} > a_{12}$, $b_{12} > b_{11}$, $b_{21} > b_{22}$. Let the players' 'traditional' behavior be 'reply best to the opponent's latest action'. Therefore, traditionally, in each round each player chooses his/her pure strategy that brings him/her the maximal benefit provided the other player repeats his/her pure strategy used in the previous round. For player 1, that behavior is easily formalized as the deterministic behavior strategy p^0 such that $p^0_{i1} = (1, 0)$, $p^0_{i2} = (0, 1)$ $(i = 1, 2)$; and for player 2 as the deterministic behavior strategy q^0 such that $q^0_{1j} = (0, 1)$, $q^0_{2j} = (1, 0)$ $(j = 1, 2)$. We call p^0 and q^0 the *best reply* strategies of players 1 and 2, respectively. The infinite repeated game corresponding to the players' best reply strategies will be called the *best reply repeated game.*

Now let us allow each player to slightly deviate from his/her 'traditional' behavior, namely, to give a small probability for choosing, in each round, his pure strategy that does not reply best to the opponent's pure strategy realized in the previous round. We call a player's behavior strategy that describes such type of behavior the player's ε-best reply strategy. More specifically, given a small $\varepsilon > 0$, we define the ε-*best reply* strategy of player 1, p, by $p_{i1} = (1-\varepsilon_1, \varepsilon_1)$, $p_{i2} = (\varepsilon_2, 1-\varepsilon_2)$ $(i = 1, 2)$ with arbitrary nonnegative $\varepsilon_1, \varepsilon_2 \leq \varepsilon$; and we define the ε-*best reply* strategy of player 2, q, by $q_{1j} = (\varepsilon_1, 1 - \varepsilon_1)$, $q_{2j} = (1 - \varepsilon_2, \varepsilon_2)$ $(j = 1, 2)$ with arbitrary nonnegative $\varepsilon_1, \varepsilon_2 \leq \varepsilon$.

Let the players' admissible behavior strategies be his/her ε-best reply strategies; in this way we define S_1 and S_2 – see section 2. We call the above defined behavior game the 2×2 ε-*best reply* one.

Note that S_1 and S_2 include the deterministic best reply strategies; therefore, S_1 and S_2 are not strictly randomized (see section 3). Consequently, for the 2×2 ε_0-best reply behavior game, the conditions sufficient for the existence of a Nash equilibrium, given in Theorem 1, do not hold. The next theorem states the existence and structure of the Nash equilibrium in the 2×2 ε_0-best reply behavior game.

Theorem 2. *Let $a_{12} \neq a_{21}$, $b_{11} \neq b_{22}$ and ε be sufficiently small. Then the following statements hold true:*

(i) the 2×2 ε-best reply behavior game has the single Nash equilibrium (p^*, q^*);

(ii) $p^*_{i1} = (1, 0)$, $p^*_{i2} = (\varepsilon, 1 - \varepsilon)$ if $a_{12} > a_{21}$, and $p^*_{i1} = (1 - \varepsilon, \varepsilon)$, $p^*_{i2} = (0, 1)$ $(i = 1, 2)$ if $a_{12} < a_{21}$ $(i = 1, 2)$;

(iii) $q^*_{1j} = (\varepsilon, 1 - \varepsilon)$, $q^*_{2j} = (1, 0)$ if $b_{11} > b_{22}$, and $q^*_{1j} = (0, 1)$, $q^*_{2j} = (1 - \varepsilon, \varepsilon)$ if $b_{11} < b_{22}$ $(j = 1, 2)$.

Thus, the equilibrium behavior strategies differ from the 'traditional' deterministic ones and have necessarily non-trivial stochastic components. A substantial interpretation can be the following: as soon as the players realize that they are allowed to introduce stochastic perturbations in their 'traditional' deterministic best reply behaviors, they get a motivation to change their 'traditional' deterministic behaviors to the equilibrium stochastic ones that are more favorable for both of them.

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From sequential analysis to optimal stopping – revisited

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1. The talk will cover three topics:

1) Sequential testing and change point detection
2) Optimal stopping of diffusions using harmonic functions
3) Determination of saddle points of stopping games

We give two examples which describe our viewpoint. Starting from the classical Wald sequential probability ratio test we elaborate a structure, which is present in many Bayes testing problems.

2. At first we consider the problem of testing the sign of the drift of Brownian motion W for the simple hypothesis H_0: $-\theta$ versus H_1: $+\theta$ with $\theta > 0$. We assume $0-1$ loss and observation cost c per unit time. The Bayes risk for the prior $\frac{1}{2}\delta_{-\theta} + \frac{1}{2}\delta_\theta$ is then defined as

$$R(T,\delta) := \frac{1}{2}\left(P_{-\theta}[H_0 \text{ rejected } (\delta)] + cE_{-\theta}T\right) + \frac{1}{2}\left(P_\theta[H_1 \text{ rejected } (\delta)] + cE_\theta T\right).$$

The goal is to find (T^*, δ^*) which minimize this risk. Let δ_T^* denote the decision rule, which rejects H_0 when $W_T > 0$. Then $R(T, \delta_T^*) \le R(T,\delta)$ holds for all decision rules δ and stopping times T. Then one can show

$$R(T,\delta_T^*) = \int h(\theta|W_T|)dQ, \qquad (*)$$

where $h(x) = \left[e^{-2x}/(1+e^{-2x})\right] + \frac{c}{\theta^2}x(1-e^{-2x})/(1+e^{-2x})$ and $Q = \frac{1}{2}P_{-\theta} + \frac{1}{2}P_\theta$.

For $x > 0$ the function h is convex and has a minimum in $b^*(c)$. Thus $R(T,\delta) \ge h(b^*(c))$. Let $T^* = \inf\{t > 0 \mid \theta|W(t)| \ge b^*(c)\}$ denote the stopping time, which stops in the minimum of h. Then (T^*, δ_T^*) minimizes the Bayes risk $(*)$.

The structure given in $(*)$ is also present when testing composite hypotheses, for certain change-point detection problems and for other testing problems with composite hypotheses. In the case of discrete observations one cannot stop in the minimum with probability 1 and one has to consider overshoot corrections.

3. The second example discusses a classical stopping problem: Let W denote Brownian motion with $M_0 = 1$. Then

$$E_{x_0}\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) = \max!$$

is to maximize over all stopping times. Let $H(x) = \int_0^\infty e^{ux-u^2/2}u^{2\beta-1}du$ with $\beta > 0$ and assume that there exists a unique point x^* with

$$\sup_{x\in\mathbb{R}} \frac{g(x)}{H(x)} = \frac{g(x^*)}{H(x^*)} = C^* \text{ and } 0 < C^* < \infty.$$

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Then $M_t = (t+1)^{-\beta} H\left(\frac{X_t}{\sqrt{t+1}}\right)/H(x_0)$ is a positive martingale with starting value 1 and further

$$E_{x_0}\left((T+1)^{-\beta} g\left(\frac{X_T}{\sqrt{T+1}}\right)\right) = H(x_o) E_{x_0} \frac{g\left(\frac{X_T}{\sqrt{T+1}}\right)}{H\left(\frac{X_T}{\sqrt{T+1}}\right)} M_T \leq H_0(x_0) C^*.$$

Let $T^* = \inf\{t > 0 \mid \frac{X_t}{\sqrt{t+1}} = x^*\}$, then if $x_0 < x^*$ it holds $P_{x_0}(T^* < \infty) = 1$ and $EM_{T^*} = 1$. Thus T^* is optimal.

For the special case: $h(x) = x$, $x_0 = 0$, and $\beta = \frac{1}{2}$ one has

$$E(X_T/(T+1)) = \max!$$

Then x^* is the solution of $x = (1-x^2)\int_0^\infty e^{-ux-u^2/2} du$, a result once derived by L. Shepp.

In general, let X denote a diffusion process and consider the problem to solve

$$V(x) = \sup_\tau E_x e^{-r\tau} g(X_\tau),$$

where x denote the starting point of X.

We suggest to find a positive function h such that $M_t = e^{-rt} h(X_t)$ is a positive local martingale and $\sup_x \frac{g}{h}(x) = C^* < \infty$. Then

$$\begin{aligned} E_x e^{-r\tau} g(X_\tau) &= E_x\left(e^{-r\tau} h(X_\tau) \frac{g(X_\tau)}{h(X_\tau)}\right) \\ &\leq C^* E_x e^{-r\tau} h(X_\tau) \\ &\leq C^* h(x). \end{aligned}$$

If we can find a stopping time τ^* with $\frac{g}{h}(X_{\tau^*}) = C^*$ and $E_x(e^{-r\tau^*} h(X_{\tau^*})) = h(x)$, then the inequalities become equalities and the optimal stopping problem has as solution $V(x) = C^* h(x)$.

We shall give several examples of this method and shall characterize with it the optimal stopping set $\{V = g\}$ in a more concrete way.

4. In the third part we consider stopping games, which can be interpreted as options in the sense of Kifer. We extend the approach described above to give sufficient conditions for Nash-equilibria of such games. This extension uses appropriate harmonic functions which are neither convex nor concave.

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Singular control and optimal stopping of SPDEs, and backward SPDEs with reflection

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1. Let B_t, $t \geq 0$ be an m-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let D be a bounded smooth domain in $\mathbb{R}^d$. Fix $T > 0$ and let $\phi(\omega, x)$ be an $\mathcal{F}_T$-measurable $H = L^2(D)$-valued random variable. Let

$$g : [0,T] \times D \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$$

be a given measurable mapping and $L(t,x) : [0,T] \times D \to \mathbb{R}$ a given continuous function. Consider the problem to find $\mathcal{F}_t$-adapted random fields $u(t,x) \in \mathbb{R}, Z(t,x) \in \mathbb{R}^m, \eta(t,x) \in \mathbb{R}^+$ left-continuous and nondecreasing w.r.t. t, such that

$$\begin{aligned} &du(t,x) = -Au(t,x)dt - g(t,x,u(t,x),Z(t,x))dt + Z(t,x)dB_t \\ &\qquad -\eta(dt,x); \quad (t,x) \in (0,T) \times D, \qquad (1) \\ &u(t,x) \geq L(t,x); \quad (t,x) \in (0,T) \times D, \\ &\int_0^T \int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0; \quad (t,x) \in (0,T) \times D, \\ &u(T,x) = \phi(x); \quad x \in D, \quad a.s. \qquad (2) \end{aligned}$$

where A is a second order linear partial differential operator. This is a *backward stochastic partial differential equation (BSPDE) with reflection*, an RBSPDE for short.

It is now well-known that the maximum principle method for solving a classical stochastic control problem for stochastic partial differential equations involves a BSPDE for the adjoint processes $p(t,x), q(t,x)$. See [8].

The purpose of this paper is threefold:
(i) We study a class of *singular* control problems for SPDEs and prove a maximum principle for the solution of such problems. This maximum principle leads to a reflected backward stochastic partial differential equation.
(ii) We study backward stochastic *partial* differential equations (BSPDEs) with reflection. As an illustration of our results we apply them to a singular optimal harvesting problem from a population whose density is modeled by a stochastic reaction-diffusion equation.
(iii) We establish a relation between RBSPDEs and optimal stopping of SPDEs, and we apply this to solve a *risk minimizing stopping problem.*

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Backward SDEs with partially nonpositive jumps and Hamilton-Jacobi-Bellman IPDEs

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We consider a class of BSDEs where the jumps component of the solution is subject to a partial nonpositive constraint. After proving existence and uniqueness of a minimal solution under mild assumptions, we give a dual representation of this solution as an essential supremum over a family of equivalent change of probability measures. We then show how minimal solutions to our BSDE class provide actually a new probabilistic representation for integro-partial differential equations (IPDEs) of Hamilton-Jacobi-Bellman (HJB) type, when dealing with a suitable Markovian framework. Joint work with I. Kharroubi.

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Extremal martingales. Stochastic optimization and optimal stopping

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Availability of market prices of call options of all strikes determines the risk-neutral distribution of the underlying asset at the terminal time. Finding the maximum and minimum price of various derivatives whose prices depend on the maximal value and the terminal value (such as barrier options) has been studied in the last 15 years or so by Hobson, Cox, Obloj, Brown, and others, and some quite complete results are known. This talk takes as its starting point some older work [1] characterizing the possible joint laws of the maximum and terminal value of a martingale; this converts the problem of finding the extremal martingale into a linear programming problem, an observation which allows effective numerical solution. I hope to be able to talk about more recent work with Moritz Duembgen characterizing the possible joint distributions of the maximum, minimum and terminal value of a continuous martingale.

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Multilevel primal and dual approaches for pricing American options

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1. Let $(Z_j)_{j\geq 0}$ be a nonnegative adapted process on a filtered probability space $(\Omega, \mathbb{F} = (\mathcal{F}_j)_{j\geq 0}, \mathbb{P})$ representing the discounted payoff of an American option, so that the holder of the option receives Z_j if the option is exercised at time $j \in \{0, \ldots, T\}$ with $T \in \mathbb{N}_+$. The pricing of American options can be formulated as a primal-dual problem. The primal representation corresponds to the following optimal stopping problems

$$Y_j^* := \max_{\tau \in \mathcal{T}[j,\ldots,T]} \mathbb{E}_{\mathcal{F}_j}[Z_\tau], \quad j = 0, \ldots, T,$$

where $\mathcal{T}[j, \ldots, T]$ is the set of $\mathbb{F}$-stopping times taking values in $\{j, \ldots, T\}$. The process $\left(Y_j^*\right)_{j\geq 0}$ is called the Snell envelope. Y^* is a supermartingale satisfying the Bellman principle

$$Y_j^* = \max\left(Z_j, \mathbb{E}_{\mathcal{F}_j}[Y_{j+1}^*]\right), \quad 0 \leq j < T, \quad Y_T^* = Z_T.$$

An exercise policy is a family of stopping times $(\tau_j)_{j=0,\ldots,T}$ such that $\tau_j \in \mathcal{T}[j, \ldots, T]$.

During the nineties the primal approach was the only method available. Some years later a quite different "dual" approach has been discovered by [8] and [5]. The next theorem summarizes their results.

Theorem 1. *Let $\mathcal{M}$ denote the space of adapted martingales, then we have the following dual representation for the value process Y_j^**

$$\begin{aligned} Y_j^* &= \inf_{\pi \in \mathcal{M}} \mathbb{E}_{\mathcal{F}_j}\left[\max_{s\in\{j,\ldots,T\}} (Z_s - \pi_s + \pi_j)\right] \\ &= \max_{s\in\{j,\ldots,T\}} (Z_s - \pi_s^* + \pi_j^*) \quad a.s., \end{aligned}$$

where

$$Y_j^* = Y_0^* + \pi_j^* - A_j^* \tag{1}$$

is the (unique) Doob decomposition of the supermartingale Y_j^*. That is, π^* is a martingale and A^* is an increasing process with $\pi_0 = A_0 = 0$ such that (1) holds.

2. Assume that we are given a stopping family (τ_j) that is *consistent,* i.e.

$$\tau_j > j \Rightarrow \tau_j = \tau_{j+1}, \quad j = 0, \ldots, T-1.$$

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The stopping policy defines a lower bound for Y^* via

$$Y_j = \mathbb{E}_{\mathcal{F}_j}[Z_{\tau_j}], \quad j = 0, \ldots, T.$$

Consider now a new family $(\widehat{\tau}_j)_{j=0,\ldots,T}$ defined by

$$\widehat{\tau}_j := \inf\left\{k : j \le k < T,\ Z_k \ge \mathbb{E}_{\mathcal{F}_k}[Z_{\tau_{k+1}}]\right\} \wedge T. \tag{2}$$

The basic idea behind (2) goes back to [6] in fact. For more general versions of policy iteration and their analysis, see [7]. Next, we introduce the $(\mathcal{F}_j)$-martingale

$$\pi_j = \sum_{k=1}^{j} \left(\mathbb{E}_{\mathcal{F}_k}[Z_{\tau_k}] - \mathbb{E}_{\mathcal{F}_{k-1}}[Z_{\tau_k}]\right), \quad j = 0, \ldots, T, \tag{3}$$

and then consider,

$$\widetilde{Y}_j := \mathbb{E}_{\mathcal{F}_j}\left[\max_{k=j,\ldots,T}(Z_k - \pi_k + \pi_j)\right],$$

along with

$$\widehat{Y}_j := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_j}], \quad j = 0, \ldots, T.$$

The following theorem states that $\widehat{Y}$ is an improvement of Y and that the Snell envelope process Y_j^* lies between $\widehat{Y}_j$ and $\widetilde{Y}_j$ with probability 1.

Theorem 2. *It holds*

$$Y_j \le \widehat{Y}_j \le Y_j^* \le \widetilde{Y}_j, \quad j = 0, \ldots, T \quad a.s.$$

3. The main issue in the Monte Carlo construction of $\widehat{Y}$ and $\widetilde{Y}$ in a Markovian environment is the estimation of the conditional expectations in (2) and (3). We thus assume that the cash-flow Z_j is of the form $Z_j = Z_j(X_j)$ for some underlying (possibly high-dimensional) Markovian process X . A canonical approach is the use of sub simulations. In this respect we consider an enlarged probability space $(\Omega, \mathbb{F}', \mathbb{P})$, where $\mathbb{F}' = (\mathcal{F}'_j)_{j=0,\ldots,T}$ and $\mathcal{F}_j \subset \mathcal{F}'_j$ for each j. On the enlarged space we consider $\mathcal{F}'_j$ measurable estimations $\mathcal{C}_{j,M}$ of $C_j = \mathbb{E}_{\mathcal{F}_j}\left[Z_{\tau_{j+1}}\right]$ as being standard Monte Carlo estimates based on M sub simulations. More precisely

$$\mathcal{C}_{j,M} = \frac{1}{M}\sum_{m=1}^{M} Z_{\tau_{j+1}^{(m)}}(X^{j,X_j}_{\tau_{j+1}^{(m)}})$$

where the $\tau_{j+1}^{(m)}$ are evaluated on M sub trajectories all starting at time j in X_j. Obviously, $\mathcal{C}_{j,M}$ is an unbiased estimator for C_j with respect to $\mathbb{E}_{\mathcal{F}_j}[\cdot]$. We thus end up with a simulation based versions of (2) and (3) respectively,

$$\widehat{\tau}_{j,M} := \inf\{k : j \le k < T,\ Z_k > \mathcal{C}_{k,M}\} \wedge T, \quad j = 0, ..., T,$$

$$\pi_{j,M} \quad := \quad \sum_{k=1}^{j} (Z_k - \mathcal{C}_{k-1,M})\, 1_{\{\tau_k = k\}} + \sum_{k=1}^{j} (\mathcal{C}_{k,M} - \mathcal{C}_{k-1,M})\, 1_{\{\tau_k > k\}}.$$

Denote

$$\widehat{Y}_{j,M} := \mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_{j,M}}], \quad j = 0, \ldots, T$$

and

$$\widetilde{Y}_{j,M} := \mathbb{E}_{\mathcal{F}_j}\left[\max_{k=j,\ldots,T}(Z_k - \pi_{k,M} + \pi_{j,M})\right].$$

Theorem 3. *Let us assume that there exist constants $B_{0,j} > 0$, $j = 0, \ldots, T-1$, and $\alpha > 0$, such that for any $\delta > 0$ and $j = 0, \ldots, T-1$,*

$$\mathbb{P}(|\mathbb{E}_{\mathcal{F}_j}[Z_{\widehat{\tau}_{j+1}}] - Z_j| \le \delta) \le B_{0,j}\delta^{\alpha}.$$

Further suppose that there are constants B_1 and B_2, such that $|Z_j| < B_1$ and

$$Var_{\mathcal{F}_j}[Z_{\tau_{j+1}}] := \mathbb{E}_{\mathcal{F}_j}[(Z_{\tau_{j+1}} - C_j)^2] < B_2, \quad a.s. \tag{4}$$

for $j = 0, \ldots, T-1$. It then holds,

$$|\widehat{Y}_0 - \widehat{Y}_{0,M}| \le M^{-\frac{1+\alpha}{2}}\, B \sum_{k=0}^{T-1} B_{0,k},$$

with some constant B depending only on α, B_1 and B_2. Moreover, if for any $\delta > 0$

$$\mathbb{P}(|\mathbb{E}_{\mathcal{F}_j}[Z_{\tau_{j+1}}] - Z_j| \le \delta) \le \overline{B}_{0,j}\delta^{\overline{\alpha}}$$

with some positive constants $\overline{\alpha}$ and $\overline{B}_{0,j}$, $j = 0, \ldots, T-1$, then

$$\mathbb{E}[(Z_{\widehat{\tau}_{0,M}} - Z_{\widehat{\tau}_0})^2] \le M^{-\overline{\alpha}/2} 2B_1^2 \overline{B} \sum_{j=0}^{T-1} \overline{B}_{0,j}.$$

Theorem 4. *Introduce for $\mathcal{Z} := \max_{j=0,\ldots,T}(Z_j - \pi_j)$, the random set*

$$\mathcal{Q} = \{j : Z_j - \pi_j = \mathcal{Z}\},$$

and the $\mathcal{F}_T$ measurable random variable

$$\Lambda := \min_{j \notin \mathcal{Q}} (\mathcal{Z} - Z_j + \pi_j),$$

with $\Lambda := +\infty$ if $\mathcal{Q} = \{0, \ldots, T\}$. Obviously $\Lambda > 0$ a.s. Further suppose that

$$\mathbb{E}[\Lambda^{-\xi}] < \infty \text{ for some } 0 < \xi \le 1, \quad \text{and} \quad \#\mathcal{Q} = 1.$$

It then holds,

$$\left|\widetilde{Y}_0 - \widetilde{Y}_{0,M}\right| \le CM^{-\frac{\xi+1}{2}}$$

for some constant C.

For a fixed natural number L and a geometric sequence $m_l = m_0\kappa^l$, for some m_0, $\kappa \in \mathbb{N}$, $\kappa \geq 2$, we consider in the spirit of [4] the telescoping sum

$$\widehat{Y}_{m_L} = \widehat{Y}_{m_0} + \sum_{l=1}^{L} \left(\widehat{Y}_{m_l} - \widehat{Y}_{m_{l-1}} \right),$$

where $\widehat{Y}_m := \widehat{Y}_{0,m}$. Next we take a set of natural numbers $\mathbf{n} := (n_0, \ldots, n_L)$ satisfying $n_0 > ... > n_L \geq 1$, and simulate an initial set of cash-flows

$$\left\{ Z^{(j)}_{\widehat{\tau}_{m_0}}, \quad j = 1, ..., n_0 \right\},$$

due to an initial set of trajectories $\{X^{0,x,(j)}_{\cdot}, j = 1, ..., n_0\}$, where

$$Z^{(j)}_{\widehat{\tau}_{m_0}} := Z_{\widehat{\tau}^{(j)}_{0,m_0}} \left(X^{0,x,(j)}_{\widehat{\tau}^{(j)}_{0,m_0}} \right).$$

Next we simulate *independently* for each level $l = 1, ..., L$, a set of pairs

$$\left\{ (Z^{(j)}_{\widehat{\tau}_{m_l}}, Z^{(j)}_{\widehat{\tau}_{m_{l-1}}}), \quad j = 1, \ldots, n_l \right\}$$

due to a set of trajectories $X^{0,x,(j)}_{\cdot}$, $j = 1, ..., n_l$, to obtain the multilevel estimator

$$\widehat{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0} \sum_{j=1}^{n_0} Z^{(j)}_{\widehat{\tau}_{m_0}} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{j=1}^{n_l} \left(Z^{(j)}_{\widehat{\tau}_{m_l}} - Z^{(j)}_{\widehat{\tau}_{m_{l-1}}} \right) \text{ for estimating } \widehat{Y}. \tag{5}$$

4. With the notations of the previous section we define

$$\widetilde{Y}_{m_L} = \widetilde{Y}_{m_0} + \sum_{l=1}^{L} [\widetilde{Y}_{m_l} - \widetilde{Y}_{m_{l-1}}],$$

where $\widetilde{Y}_m := \widetilde{Y}_{0,m}$. Given a sequence $\mathbf{n} = (n_0, \ldots, n_L)$ with $1 \leq n_0 < \ldots < n_L$, we then simulate for $l = 0$ an initial set of trajectories

$$\left\{ (Z^{(i)}_j, \pi^{(i)}_{j,m_0}), \quad i = 1, ..., n_0, \quad j = 0, \ldots, T, \right\}$$

of the two-dimensional vector process (Z_j, π_{j,m_0}), and then for each level $l = 1, \ldots, L$, *independently,* a set of trajectories

$$\left\{ (Z^{(i)}_j, \pi^{(i)}_{j,m_{l-1}}, \pi^{(i)}_{j,m_l}), \quad i = 1, ..., n_l, \quad j = 0, \ldots, T \right\}$$

of the vector process $(Z_j, \pi_{j,m_{l-1}}, \pi_{j,m_l})$. Based on this simulation we consider the following multilevel estimator:

$$\widetilde{\mathcal{Y}}_{\mathbf{n},\mathbf{m}} := \frac{1}{n_0} \sum_{i=1}^{n_0} \mathcal{Z}^{(i)}_{m_0} + \sum_{l=1}^{L} \frac{1}{n_l} \sum_{i=1}^{n_l} [\mathcal{Z}^{(i)}_{m_l} - \mathcal{Z}^{(i)}_{m_{l-1}}] \tag{6}$$

with $\mathcal{Z}_{m_l}^{(i)} := \max_{j=0,\dots,T}\left(Z_j^{(i)} - \pi_{j,m_l}^{(i)}\right)$, $i = 1,\dots,n_l$, $l = 0,\dots,L$.

5. We now consider the numerical complexity of the multilevel estimators (5) and (6), for convenience generically denoted by $X_{\mathbf{n},\mathbf{m}}$. Assume that there are some positive constants γ, β, μ_∞, σ_∞ and $\mathcal{V}_\infty$ such that $\mathrm{Var}[\mathcal{X}_m] \le \sigma_\infty^2$,

$$|X - \mathbb{E}[\mathcal{X}_m]| \le \mu_\infty m^{-\gamma}, \quad m \in \mathbb{N} \quad \text{and} \tag{7}$$

$$\mathbb{E}[\mathcal{X}_{m_l} - \mathcal{X}_{m_{l-1}}]^2 \le \mathcal{V}_\infty m_l^{-\beta}, \quad l = 1,\dots,L. \tag{8}$$

Theorem 5. *Let us assume that* $0 < \beta \le 1$, $\gamma \ge \frac{1}{2}$ *and* $m_l = m_0\kappa^l$ *for some fixed* κ *and* $m_0 \in \mathbb{N}$. *Fix some* $0 < \epsilon < 1$. *Let* $L = L(\epsilon)$ *be the integer part of*

$$\gamma^{-1}\ln^{-1}\kappa \ln\left[\frac{\sqrt{2}\mu_\infty}{m_0^\gamma \epsilon}\right], \quad \textit{and} \quad n_l = n_0\kappa^{-l(1+\beta)/2} \quad \textit{with}$$

$$n_0 = n_0(\epsilon) = \frac{2\sigma_\infty^2}{\epsilon^2} + \frac{2\mathcal{V}_\infty}{\epsilon^2 m_0^\beta}\frac{\kappa^{L(1-\beta)/2} - 1}{\kappa^{(1-\beta)/2} - 1}\kappa^{(1-\beta)/2}.$$

Then the complexity needed to achieve the accuracy $\varepsilon := \sqrt{\mathbb{E}[(X - X_{\mathbf{n},\mathbf{m}})^2]} < \varepsilon$ *is*

$$\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon) = O(\epsilon^{-2-\frac{1-\beta}{\gamma}}), \quad \epsilon \searrow 0, \quad \textit{for } \beta < 1,$$

$$\mathcal{C}_{ML}^{\mathbf{n},\mathbf{m}}(\epsilon) = O(\epsilon^{-2}\ln^2\epsilon), \quad \epsilon \searrow 0, \quad \textit{for } \beta = 1.$$

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Asymptotically optimal discretization of hedging strategies with jumps

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1. A basic problem in mathematical finance is to replicate a random claim with $\mathcal{F}_T$-measurable payoff H_T with a portfolio involving only the underlying asset Y and cash. When Y follows a diffusion process of the form

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t, \tag{1}$$

it is known that under minimal assumptions, a random payoff depending only on the terminal value of the asset $H_T = H(Y_T)$ can be replicated with the so-called delta hedging strategy. However, to implement such a strategy, the hedging portfolio must be readjusted continuously, which is of course physically impossible and anyway irrelevant because of the presence of microstructure effects and transaction costs. For this reason, the optimal strategy is always replaced with a piecewise constant one, leading to a discretization error. The relevant question is then to find the optimal discretization dates. Of course, it is intuitively clear that readjusting the portfolio at regular deterministic intervals is not optimal. However, the optimal strategy for fixed n is very difficult to compute.

Fukasawa [1] simplifies this problem by assuming that the hedging portfolio is readjusted at high frequency. The performance of different discretization strategies can then be compared based on their asymptotic behavior as the number of readjustment dates n tends to infinity, rather than the performance for fixed n. Consider a discretization rule : a sequence of discretization strategies

$$0 = T_0^n < T_1^n < \cdots < T_j^n < \ldots,$$

with $\sup_j |T_{j+1}^n - T_j^n| \to 0$ as $n \to \infty$ and let $N_T^n := \max\{j \geq 0; T_j^n \leq T\}$ be the total number of readjustment dates on the interval $[0, T]$ for given n. To compare different discretization rules in terms of their asymptotic behavior, Fukasawa [1] uses the criterion

$$\lim_{n\to\infty} E[N_T^n]E[\langle \mathcal{E}^n \rangle_T], \tag{2}$$

where $\langle \mathcal{E}^n \rangle$ is the quadratic variation of the semimartingale $(\mathcal{E}_t^n)_{t\geq 0}$. He finds that when the underlying asset is a continuous semimartingale, the functional (2) admits a nonzero lower bound over all discretization rules, and exhibits a specific explicit rule based on hitting times which attains this lower bound and is therefore called *asymptotically efficient.*

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While the above approach is quite natural and provides very explicit results, it fails to take into account important factors of market reality. First, the asymptotic functional (2) is somewhat ad hoc, and does not reflect any specific model for the transaction costs. Second, the continuity assumption, especially at relatively high frequencies, is not realistic.

The objective of our work is therefore two-fold. First, we develop a framework for characterizing the asymptotic efficiency of discretization strategies which takes into account the transaction costs. Second, we remove the continuity assumption in order to understand the effect of the activity of small jumps (often quantified by the Blumenthal-Getoor index) on the optimal discretization strategies.

2. Our goal is to study and compare different discretization rules for the stochastic integral

$$\int_0^T X_{t-} dY_t,$$

where X and Y are semimartingales with jumps. More precisely, our principal assumptions on the processes X and Y are

- The process Y is a $\mathbb{F}$-local martingale, whose predictable quadratic variation satisfies $\langle Y \rangle_t = \int_0^t A_s ds$, where the process (A_t) is càdlàg and locally bounded.
- The process X is a pure jump semimartingale defined via the stochastic representation

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \int_{|z|\le 1} z(M-\mu)(ds \times dz) + \int_0^t \int_{|z|>1} zM(ds \times dz), \tag{3}$$

  where M is the jump measure of X and μ is its predictable compensator, which satisfies additionally $\mu(dt \times dz) = dt \times \lambda_t K_t(z)\, \nu(dz)$, where λ is a positive càdlàg process, K is a random function which is in some sense "close to 1" when z is close to 0 and ν is a Lévy measure satisfying

$$x^\alpha \nu((x,\infty)) \to c_+ \quad \text{and} \quad x^\alpha \nu((-\infty,-x)) \to c_- \quad \text{when} \quad x \to 0. \tag{H_α}$$

  for some $\alpha \in (1,2)$ and constants $c_+ \ge 0$ and $c_- \ge 0$ with $c_+ + c_- > 0$.

A *discretization rule* is a family of stopping times $(T_i^\varepsilon)_{i\ge 0}^{\varepsilon>0}$ parameterized by a nonnegative integer i and a positive real ε, such that for every $\varepsilon > 0$, $0 = T_0^\varepsilon < T_1^\varepsilon < T_2^\varepsilon < \ldots$, and $\sup\{i : T_i^\varepsilon \le T\} < \infty$. For a fixed discretization rule and a fixed ε, we denote $\eta(t) = \sup\{T_i^\varepsilon : T_i^\varepsilon \le t\}$ and $N_T = \sup\{i : T_i^\varepsilon \le T\}$.

The performance of a discretization rule is measured by the error functional $\mathcal{E}(\varepsilon) : (0,\infty) \to [0,\infty)$ and the cost functional $\mathcal{C}(\varepsilon) : (0,\infty) \to [0,\infty)$. We consider the error functional given by the L^2 error

$$\mathcal{E}(\varepsilon) := E\left[\left(\int_0^T (X_{t-} - X_{\eta(t)-}) dY_t\right)^2\right] \tag{4}$$

and a family of cost functionals of the form

$$\mathcal{C}^\beta(\varepsilon) = E\left[\sum_{i\geq 1: T_i^\varepsilon \leq T} |X_{T_i^\varepsilon} - X_{T_{i-1}^\varepsilon}|^\beta\right]. \tag{5}$$

The case $\beta = 0$ correspond to a fixed cost for each discretization point, and the case $\beta = 1$ corresponds to proportional costs.

In our framework, a discretization strategy will be said to be asymptotically optimal for a given cost functional if no strategy has (asymptotically, for large costs) a lower discretization error and a smaller cost.

Motivated by the form of the explicit asymptotically optimal strategy found by Fukasawa [1] and the readjustment rules used by market practitioners, we consider discretizations based on the hitting times of the process X. Such a discretization rule is defined by a pair of positive $\mathbb{F}$-adapted càdlàg processes $(\overline{a}_t)_{t\geq 0}$ and $(\underline{a}_t)_{t\geq 0}$. The discretization dates are then given by

$$T_{i+1}^\varepsilon = \inf\{t > T_i^\varepsilon : X_t \notin (X_{T_i^\varepsilon} - \varepsilon \underline{a}_{T_i^\varepsilon}, X_{T_i^\varepsilon} + \varepsilon \overline{a}_{T_i^\varepsilon})\}.$$

3. We characterize explicitly the asymptotic behavior of the errors and costs associated to our random discretization rules, by showing that, under suitable assumptions,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathcal{E}(\varepsilon) = E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right] \tag{6}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-\beta}\mathcal{C}^\beta(\varepsilon) = E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right], \tag{7}$$

where, for $\underline{a}, \overline{a} \in (0, \infty)$,

$$f(\underline{a}, \overline{a}) = E\left[\int_0^{\tau^*} (X_t^*)^2 dt\right], \quad g(\underline{a}, \overline{a}) = E[\tau^*] \quad \text{and} \quad u^\beta(\underline{a}, \overline{a}) = E[|X_{\tau^*}^*|^\beta] < \infty.$$

with $\tau^* = \inf\{t \geq 0 : X_t^* \notin (-\underline{a}, \overline{a})\}$, where X^* is a strictly α-stable process with Lévy density

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}.$$

The above result allows to prove that we may look for optimal barriers $\underline{a}$ and $\overline{a}$ as minimizers of

$$\min\left\{A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} + c\lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\right\}. \tag{8}$$

The parameter c may be chosen by the trader depending on the maximum acceptable cost: the bigger c, the smaller will be the cost of the strategy and, consequently the bigger its error. The functions f, g and u appearing above must in general be

computed numerically, however, in the case when the limiting process X^* is a symmetric stable process, which corresponds for example to the CGMY model very popular in practice, the results are completely explicit, as will be shown in the next paragraph.

4. Assume that Y is an exponential of a Lévy process: $Y_t = e^{Z_t}$ where Z is a Lévy process without diffusion part, and whose Lévy measure has a density $\nu(x)$ satisfying $\nu(x) \sim \frac{c}{|x|^{1+\alpha}}$, $x \to 0$. Then $A_t = \Sigma Y_t^2$ with $\Sigma = \int (e^z - 1)^2 \nu(z) dz$. The quadratic hedging strategy in this case has been given by several authors and is known to have a Markov structure: $X_t = \phi(t, Y_t)$ for a deterministic function ϕ. In this case we may compute

$$\lambda_t = \left(Y_t \frac{\partial \phi(t, Y_t)}{\partial Y} \right)^{\alpha}$$

and therefore

$$a_t = c \left(\frac{\partial \phi(t, Y_t)}{\partial Y} \right)^{\frac{\alpha}{2+\alpha-\beta}} Y_t^{\frac{\alpha-2}{\alpha-\beta+2}}.$$

When $\beta = 0$ and $\alpha \to 2$, we find that the optimal size of the rebalancing interval is proportional to the square root of $\frac{\partial \phi(t, Y_t)}{\partial Y}$ (the gamma), which is consistent with the results of Fukasawa [1], quoted above. In the general case, we obtain an explicit representation for the optimal discretization dates, which includes two "tuning" parameters: the index β which determines the effect of transaction costs (fixed, proportional, etc.) and the Blumenthal-Getoor index α measuring the activity of small jumps.

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Sequential Hypothesis Testing and Changepoint Detection: Past and Future

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1. Nearly Optimal Sequential Tests of Composite Hypotheses. Let X_1, $X_2, \dots$ be a sequence of independent and identically distributed (iid) observations, and let $p_\theta(x)$ be a density (parametrized by a parameter θ) with respect to some non-degenerate sigma-finite measure μ. In his 1947 book, Wald [9, Section 6] suggested two approaches for modifying the *Sequential Probability Ratio Test* (SPRT) to test a simple null hypothesis $H_0 : \theta = \theta_0$ against a composite alternative $H_1 : \theta \in \Theta_1$. One method is to replace the likelihood ratio (LR) $\Lambda_n^\theta = \prod_{k=1}^n [p_\theta(X_k)/p_{\theta_0}(X_k)]$ by a weighted LR $\bar{\Lambda}_n = \int_{\Theta_1} w(\theta)\Lambda_n^\theta \,\mathrm{d}\theta$, using a suitably selected weight function $w(\theta)$ on the hypothesis H_1. This leads to the *Weighted SPRT* (WSPRT) $\bar{\delta} = (\bar{T}, \bar{d})$ with the stopping time $\bar{T}(A_0, A_1) = \inf\left\{n \geq 1 : \bar{\Lambda}_n \notin (A_0, A_1)\right\}$, $0 < A_0 < 1$, $A_1 > 1$. The weighted-based tests are also often called *mixture-based* tests of simply *mixtures*. The other way is to apply the generalized likelihood ratio (GLR) approach of classical fixed-sample size theory, employing the GLR statistic $\hat{\Lambda}_n = \sup_{\theta\in\Theta_1} \Lambda_n^\theta$ in place of the LR Λ_n^θ with *a priori* fixed parameters, which leads to the *Generalized Sequential Likelihood Ratio Test* (GSLRT) $\hat{\delta} = (\widehat{T}, \hat{d})$ with the stopping time $\widehat{T}(A_0, A_1) = \inf\left\{n \geq 1 : \hat{\Lambda}_n \notin (A_0, A_1)\right\}$.

In a more general case where the null hypothesis is also composite, $H_0 : \theta \in \Theta_0$, Wald [9] proposed to exploit the WSPRT with the weighted LR

$$\bar{\Lambda}_n = \frac{\int_{\Theta_1} w_1(\theta) \prod_{k=1}^n p_\theta(X_k) \,\mathrm{d}\theta}{\int_{\Theta_0} w_0(\theta) \prod_{k=1}^n p_\theta(X_k) \,\mathrm{d}\theta}.$$

Changing the measures and applying the Wald likelihood ratio identity, we obtain the upper bounds on the average error probabilities: $\bar{\alpha}_0(\bar{\delta}) = \int_{\Theta_0} \mathsf{P}_\theta(\bar{d} = 1) w_0(\theta) \,\mathrm{d}\theta \leq 1/A_1$, $\bar{\alpha}_1(\bar{\delta}) = \int_{\Theta_1} \mathsf{P}_\theta(\bar{d} = 0) w_1(\theta) \,\mathrm{d}\theta \leq A_0$. Clearly, for practical purposes, one would strongly prefer to upper-bound not the average error probabilities, which depend on a particular choice of weights, but rather the maximal error probabilities of Type I and Type II, i.e., to consider the class of tests $\mathbf{C}(\alpha_0, \alpha_1) = \{\delta : \sup_{\theta\in\Theta_0} \mathsf{P}_\theta(d = 1) \leq \alpha_0, \sup_{\theta\in\Theta_1} \mathsf{P}_\theta(d = 0) \leq \alpha_1\}$, $\alpha_0 + \alpha_1 < 1$. However, in general it is not clear how to obtain the upper bounds on maximal error probabilities of the WSPRT and the GSLRT. In this respect, the tests that are based on one-stage delayed estimators, for the first time suggested by Robbins and Siegmund [6, 7] in the context of power one tests in the beginning of seventies, represent a useful alternative considered below.

More generally, consider the following continuous- or discrete-time scenario with multiple composite hypotheses. Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathsf{P}_\theta)$, $t \in \mathbb{Z}_+ = \{0, 1, \dots\}$ or $t \in$

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$\mathbb{R}_+ = [0, \infty)$, be a filtered probability space with standard assumptions about monotonicity and, in the continuous time case $t \in \mathbb{R}_+$, also right-continuity of the σ-algebras $\mathscr{F}_t$. The parameter $\theta = (\theta_1, \ldots, \theta_\ell)$ belongs to a subset $\tilde{\Theta}$ of ℓ-dimensional Euclidean space $\mathbb{R}_\ell$. The sub-σ-algebra $\mathscr{F}_t = \mathscr{F}_t^X = \sigma(\mathbf{X}_0^t)$ of $\mathscr{F}$ is generated by the stochastic process $\mathbf{X}_0^t = \{X(u),\ 0 \le u \le t\}$ observed up to time t . The hypotheses to be tested are "$H_i:\ \theta \in \Theta_i$", $i = 0, 1, \ldots, N$ ($N \ge 1$), where Θ_i are disjoint subsets of $\tilde{\Theta}$. We will also suppose that there is an *indifference* zone $\mathrm{I}_{\mathrm{in}} \in \tilde{\Theta}$ in which there are no constraints on the probabilities of errors imposed. The indifference zone, where any decision is acceptable, is usually introduced keeping in mind that the correct action is not critical and often not even possible when the hypotheses are too close, which is perhaps the case in most, if not all, practical applications. However, in principle I_{in} may be an empty set. The probability measures P_θ and $\mathsf{P}_{\tilde{\theta}}$ are assumed to be locally mutually absolutely continuous, i.e., the restrictions P_θ^t and $\mathsf{P}_{\tilde{\theta}}^t$ of these measures to the sub-σ-algebras $\mathscr{F}_t$ are equivalent for all $0 \le t < \infty$ and all $\theta, \tilde{\theta} \in \tilde{\Theta}$.

A multihypothesis sequential test δ consists of the pair (T, d), where T is a stopping time with respect to the filtration $\{\mathscr{F}_t\}_{t \ge 0}$, and $d = d_T(\mathbf{X}_0^T) \in \{0, 1, \ldots, N\}$ is an $\mathscr{F}_T$-measurable (terminal) decision rule specifying which hypothesis is to be accepted once observations have stopped (the hypothesis H_i is accepted if $d = i$ and rejected if $d \ne i$, i.e., $\{d = i\} = \{T < \infty,\ \delta \text{ accepts } H_i\}$). The quality of a sequential test is judged on the basis of its error probabilities and expected sample sizes (or more generally on the moments of the sample size). Let $\alpha_{ij}(\delta, \theta) = \mathsf{P}_\theta(d = j) 1_{\{\theta \in \Theta_i\}}$ ($i \ne j,\ i, j = 0, 1, \ldots, N$) be the probability of accepting the hypothesis H_j by the test δ when the true value of the parameter θ is fixed and belongs to the subset Θ_i. Introduce the class of tests $\mathbf{C}(||\alpha_{ij}||) = \left\{\delta : \sup_{\theta \in \Theta_i} \alpha_{ij}(\delta, \theta) \le \alpha_{ij},\ i, j = 0, 1, \ldots, N,\ i \ne j\right\}$ for which maximal error probabilities do not exceed the given numbers α_{ij}.

While almost all results hold for continuous time too, we will focus only on the discrete time scenario. Let $\hat{\theta}_n = \hat{\theta}_n(X_1, \ldots, X_n)$ be an estimator of θ. If in density $p_\theta(X_k)$ for the kth observation we replace the parameter by the estimate $\hat{\theta}_{k-1}$ built upon the sample $(X_1, \ldots, X_{k-1})$ that includes $k-1$ observations, then $p_{\hat{\theta}_{k-1}}(X_k)$ is still a viable probability density, in contrast to the case of the GLR approach where $p_{\hat{\theta}_n}(X_k)$ is not a probability density anymore for $k \le n$. Therefore, the statistic

$$\Lambda_n^*(\theta_i) = \prod_{k=1}^{n} \frac{p_{\hat{\theta}_{k-1}}(X_k)}{p_{\theta_i}(X_k)} = \Lambda_{n-1}^*(\theta_i) \times \frac{p_{\hat{\theta}_{n-1}}(X_n)}{p_{\theta_i}(X_n)} \tag{1}$$

is a viable likelihood ratio, and it is the nonnegative P_{θ_i}-martingale with unit expectation, since $\mathsf{E}_{\theta_i}[\Lambda_n^*(\theta_i) | \mathbf{X}_1^{n-1}] = \Lambda_{n-1}^*(\theta_i)$. Therefore, one can use Wald's likelihood ratio identity for finding bounds on error probabilities if $\Lambda_n^*(\theta_i)$ is used instead of the LR with the true parameter value θ. Because of exactly this very convenient property as well as of the simple recursive structure (1) the hypothesis tests based on the adaptive LR's with one-stage delayed estimators represent a very attractive alternative to the GLR tests as well to the mixture-based tests. De-

fine the statistics $\Lambda_n^*(\Theta_i) = \prod_{k=1}^n p_{\hat{\theta}_{k-1}}(X_k)/\sup_{\theta\in\Theta_i}\prod_{k=1}^n p_\theta(X_k)$, $i = 0, 1, \ldots, N$. The multihypothesis test, which we will refer to as the *Multihypothesis Adaptive Sequential Likelihood Ratio Test* (MASLRT), has the form

$$\text{stop at the first } n \geq 1 \text{ such that for some } i \quad \Lambda_n^*(\Theta_j) \geq A_{ji} \quad \text{for all } j \neq i$$

and accept the (unique) H_i that satisfies these inequalities.

Write $\alpha_{ij}^*(\theta) = \mathsf{P}_\theta(d^* = j)1_{\{\theta\in\Theta_i\}}$ for the error probabilities of the MASLRT. It can be shown that $\sup_{\theta\in\Theta_i}\alpha_{ij}^*(\theta) \leq 1/A_{ij}$, $i \neq j$, so that $A_{ij} = 1/\alpha_{ij}$ implies $\delta^* \in \mathbf{C}(||\alpha_{ij}||)$.

For $r > 0$, the random variable ξ_n is said to converge P-*r-quickly* to a constant C if $\mathsf{E}L_\varepsilon^r < \infty$ for all $\varepsilon > 0$, where $L_\varepsilon = \sup\{n : |\xi_n - C| > \varepsilon\}$ ($\sup\varnothing = 0$).

Write $\lambda_n(\theta, \tilde{\theta}) = \log\frac{\mathrm{d}\mathsf{P}_\theta^n}{\mathrm{d}\mathsf{P}_{\tilde{\theta}}^n} = \sum_{k=1}^n \log\frac{p_\theta(X_k|\mathbf{X}_1^{k-1})}{p_{\tilde{\theta}}(X_k|\mathbf{X}_1^{k-1})}$ for the log-likelihood ratio (LLR) process. Assume that there exist positive and finite numbers $I(\theta, \tilde{\theta})$ such that

$$\frac{1}{n}\lambda_n(\theta, \tilde{\theta}) \xrightarrow[n\to\infty]{\mathsf{P}_\theta-r-\text{quickly}} I(\theta, \tilde{\theta}) \quad \text{for all} \quad \theta, \tilde{\theta} \in \Theta, \theta \neq \tilde{\theta}. \tag{2}$$

In addition, we certainly need some conditions on the behavior of the estimate $\hat{\theta}_n$ for large n, which should converge to the true value θ in a proper way. To this end, we require the following condition on the adaptive LLR process:

$$\frac{1}{n}\log\Lambda_n^*(\tilde{\theta}) \xrightarrow[n\to\infty]{\mathsf{P}_\theta-r-\text{quickly}} I(\theta, \tilde{\theta}) \quad \text{for all} \quad \theta, \tilde{\theta} \in \Theta, \theta \neq \tilde{\theta}, \tag{3}$$

so that the normalized by n LLR tuned to the true parameter value and its adaptive version converge to the same constants. In certain cases, but not always, conditions (2) and (3) imply the following conditions

$$\frac{1}{n}\log\Lambda_n^*(\Theta_i) \xrightarrow[n\to\infty]{\mathsf{P}_\theta-r-\text{quickly}} I_i(\theta) \quad \text{for all } \theta \in \Theta \setminus \Theta_i, \ i = 0, 1, \ldots, N, \tag{4}$$

where $I_i(\theta) = \inf_{\tilde{\theta}\in\Theta_i} I(\theta, \tilde{\theta})$ is assumed to be positive for all i. Let

$$J_i(\theta) = \min_{\substack{0\leq j\leq N\\ j\neq i}}[I_j(\theta)/c_{ji}] \quad \text{for} \quad \theta \in \Theta_i, \qquad J(\theta) = \max_{0\leq i\leq N} J_i(\theta) \quad \text{for} \quad \theta \in \mathrm{I}_{\mathrm{in}},$$

where $c_{ij} = \lim_{\alpha_{\max}\to 0}|\log\alpha_{ij}|/|\log\alpha_{\max}|$, $\alpha_{\max} = \max_{i,j}\alpha_{ij}$.

The following theorem establishes uniform asymptotic optimality of the MASLRT in the general non-iid case with respect to moments of the stopping time distribution. The proof is based on the technique developed by Tartakovsky [8] for multiple simple hypotheses.

Theorem 1 (MASLRT asymptotic optimality)**.** *Assume that r-quick convergence conditions* (2) *and* (4) *are satisfied. If the thresholds A_{ij} are so selected that $\sup_{\theta\in\Theta_i}\alpha_{ij}^*(\theta) \leq \alpha_{ij}$ and $\log A_{ij} \sim \log(1/\alpha_{ij})$, in particular $A_{ij} = 1/\alpha_{ij}$, then for $m \leq r$ as $\alpha_{\max} \to 0$*

$$\inf_{\delta\in\mathbf{C}(||\alpha_{ij}||)} \mathsf{E}_\theta T^m \sim \mathsf{E}_\theta[T^*]^m \sim \begin{cases} [|\log\alpha_{\max}|/J_i(\theta)]^m & \textit{for all } \theta \in \Theta_i \textit{ and } i = 0, 1, \ldots, N \\ [|\log\alpha_{\max}|/J(\theta)]^m & \textit{for all } \theta \in \mathrm{I}_{\mathrm{in}}. \end{cases}$$

Consequently, the MASLRT minimizes asymptotically the moments of the sample size up to order r uniformly in $\theta \in \Theta$ in the class of tests $\mathbf{C}(||\alpha_{ij}||)$.

This theorem generalizes previous results of Pavlov [3] and Dragalin and Novikov [1] restricted to iid exponential families, and also provides alternative conditions in iid cases that can be often easily checked. Indeed, for a multidimensional exponential family, conditions (2) are satisfied for all $r > 0$ with $I(\theta, \tilde{\theta}) = \mathsf{E}_\theta \lambda_1(\theta, \tilde{\theta})$ being the Kullback–Leibler information numbers. Also, in many particular cases, conditions (4) also hold when $\hat{\theta}_n$ is the maximum likelihood estimator (MLE). For example, assume that $X_n \sim \mathcal{N}(\mu, \sigma^2)$, $n = 1, 2, \dots$ are iid normal random variables with unknown mean μ and unknown variance σ^2 and the hypotheses are "$H_0 : \mu = 0, \sigma^2 > 0$" and "$H_1 : \mu \geq \mu_1, \sigma^2 > 0$", where μ_1 is a given positive number. In this case, $N = 1$, $\theta = (\mu, \sigma^2)$ and the variance σ^2 is a nuisance parameter. It can be verified that if $(\hat{\mu}_n, \hat{\sigma}_n^2)$ is the MLE, $\hat{\mu}_n = \max\{0, n^{-1} \sum_{k=1}^n X_k\}$, $\hat{\sigma}_n^2 = n^{-1} \sum_{k=1}^n (X_k - \hat{\mu}_n)^2$, then conditions (4) hold for all $r > 0$ with $I_1(q) = \frac{1}{2} \log[1 + (q_1 - q)^2]$, $0 \leq q < q_1$ and $I_0(q) = \frac{1}{2} \log(1 + q^2)$, $q \geq 0$, where $q = \mu/\sigma$ and $q_1 = \mu_1/\sigma$. Therefore, by Theorem 1, the ASLRT minimizes (asymptotically) all positive moments of the sample size.

2. Sequential Changepoint Detection. Assume $X_1, X_2, \dots$ is a sequence of independent observations and $X_1, \dots, X_\nu$ have density $p_{\theta_0}(x)$ while at time ν something happens and $X_{\nu+1}, X_{\nu+2}, \dots$ have density $p_\theta(x)$, $\theta \in \Theta$, $\theta_0 \notin \Theta$. The pre-change parameter θ_0 is known, but the time of change $\nu \in \{0, 1, \dots\}$ and the post-change parameter θ are unknown. Let $W(\theta)$ be a weight (mixing prior distribution) and consider the following mixture-based Shiryaev–Roberts changepoint detection procedure

$$\mathsf{T}_{\mathsf{SR}}(A) = \inf \left\{ n \geq 1 : \int_\Theta R_n^\theta W(\mathrm{d}\theta) \geq A \right\}, \quad A > 0,$$

where $R_n^\theta = \sum_{k=1}^n \prod_{i=k}^n \frac{p_\theta(X_i)}{p_{\theta_0}(X_k)}$. We refer to this procedure as the WSR.

Let E_ν^θ denote expectation with respect to the probability measure P_ν^θ when the changepoint is ν and the post-change parameter is θ and let E_∞ denote expectation when there is no change. Let $\mathsf{ARL}(T) = \mathsf{E}_\infty T$ stand for the average run length (mean time) to false alarm. Let $\lambda_n^\theta = \sum_{k=1}^n \log \frac{p_\theta(X_k)}{p_{\theta_0}(X_k)}$ be the LLR and define the conditional expected Kullback–Leibler information $\mathcal{J}_\nu^\theta(T) := \mathsf{E}_\nu^\theta(\lambda_T^\theta - \lambda_\nu^\theta | T > \nu) = I_\theta \mathsf{E}_\nu^\theta(T - \nu | T > \nu)$, where $I_\theta = \mathsf{E}_0^\theta \lambda_1^\theta$. Then the maximal Kullback–Leibler information (over both ν and θ) is

$$\sup_{\theta \in \Theta} \sup_{\nu \geq 0} \mathcal{J}_\nu^\theta(T) = \sup_{\theta \in \Theta} \left[I_\theta \sup_{\nu \geq 0} \mathsf{E}_\nu^\theta(T - \nu | T > \nu) \right].$$

If p_θ belongs to the ℓ-dimensional exponential family, then the following two results can be established. First, the WSR procedure that starts off at zero ($R_0^\theta = 0$) is second order asymptotically optimal for any mixing distribution $W(\theta)$ with

continuous density in the class $\mathbf{C}(\gamma) = \{T : \mathsf{ARL}(T) \geq \gamma\}$:

$$\sup_{\theta\in\Theta,\nu\geq 0} \mathcal{J}_\nu^\theta(\mathsf{T}_{\mathsf{SR}}) = \inf_{T\in\mathbf{C}(\gamma)} \sup_{\theta\in\Theta,\nu\geq 0} \mathcal{J}_\nu^\theta(T) + O(1) \quad \text{as} \quad \gamma \to \infty,$$

where $O(1)$ is bounded as $\gamma \to \infty$. More importantly, if the WSR procedure starts off at a specially designed point and the mixing distribution $W = W^*$ is selected also in a special way depending on the average overshoot in the one-sided SPRT, then this specially designed WSR procedure $\mathsf{T}_{\mathsf{SR}}^*$ becomes third-order asymptotically optimal, i.e.,

$$\sup_{\theta\in\Theta,\nu\geq 0} \mathcal{J}_\nu^\theta(\mathsf{T}_{\mathsf{SR}}^*) = \inf_{T\in\mathbf{C}(\gamma)} \sup_{\theta\in\Theta,\nu\geq 0} \mathcal{J}_\nu^\theta(T) + o(1) \quad \text{as} \quad \gamma \to \infty,$$

where $o(1) \to 0$ as $\gamma \to \infty$.

The proofs are based on the works by Pollak [5, 4] and recent results of Fellouris and Tartakovsky [2].

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Optimal trade execution and price manipulation in order books with time-varying liquidity

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1. *Summary:* In financial markets liquidity is not constant over time but exhibits strong seasonal patterns. We consider a limit order book model that allows for time-dependent, deterministic depth and resilience of the book and determine optimal portfolio liquidation strategies. In a first model variant we propose a trading dependent spread that increases when market orders are matched against the order book. In this model no price manipulation occurs and the optimal strategy is of the "wait region – buy region" type often encountered in singular control problems. In a second model we assume that there is no spread in the order book. Under this assumption we find that price manipulation can occur, depending on the model parameters. Even in the absence of classical price manipulation there may be transaction-triggered price manipulation. In specific cases, we can state optimal strategies in closed form. The talk is based on [8].

2. Empirical investigations have demonstrated that liquidity varies over time. In particular deterministic time-of-day and day-of-week liquidity patterns have been found in most markets. In spite of these findings the academic literature on optimal trade execution usually assumes constant liquidity during the trading time horizon. In the talk we relax this assumption and analyze the effects of deterministically varying liquidity on optimal trade execution for a risk-neutral investor. We characterize optimal strategies in terms of a trade region and a wait region and find that optimal trading strategies depend on the expected pattern of time-dependent liquidity. In the case of extreme changes in liquidity, it can even be optimal to entirely refrain from trading in periods of low liquidity. Incorporating such patterns in trade execution models can hence lower transaction costs.

Time-dependent liquidity can potentially lead to price manipulation. In periods of low liquidity, a trader could buy the asset and push market prices up significantly; in a subsequent period of higher liquidity, he might be able to unwind this long position without depressing market prices to their original level, leaving the trader with a profit after such a round trip trade. In reality such round trip trades are often not profitable due to the bid-ask spread: once the trader starts buying the asset in large quantities, the spread widens, resulting in a large cost for the trader when unwinding the position. We propose a model with trading-dependent spread and demonstrate that price manipulation does not exist in this model in spite of time-dependent liquidity. In a similar model with fixed zero spread we find that price manipulation or transaction-triggered price manipulation (a term recently

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coined by [2] and [9]) can be a consequence of time-dependent liquidity.

Our liquidity model is based on the limit order book market model of [11], which models both depth and resilience of the order book explicitly. The instantaneously available liquidity in the order book is described by the depth. Market orders issued by the large investor are matched with this liquidity, which increases the spread. Over time, incoming limit orders replenish the order book and reduce the spread; the speed of this process is determined by the resilience. We generalize the model of [11] in that both depth and resilience can be time dependent. In relation to the problem of optimal trade execution we show that there is a time dependent optimal ratio of remaining order size to bid-ask spread: If the actual ratio is larger than the optimal ratio, then the trader is in the "trade region" and it is optimal to reduce the ratio by executing a part of the total order. If the actual ratio is smaller than the optimal ratio, then the trader is in the "wait region" and it is optimal to wait for the spread to be reduced by future incoming limit orders before continuing to trade. We will see that allowing for time-varying liquidity parameters brings qualitatively new phenomena into the picture. For instance, it can happen that it is optimal to wait regardless of how big the remaining position is, while this cannot happen in the framework of [11].

Building on empirical investigations of the market impact of large transactions, a number of theoretical models of illiquid markets have emerged. One part of these market microstructure models focuses on the underlying mechanisms for illiquidity effects, e.g., [10] and [7]. We follow a second line that takes the liquidity effects as given and derives optimal trading strategies within such a stylized model market. Two broad types of market models have been proposed for this purpose. First, several models assume an instantaneous price impact, e.g., [5], [4] and [3]. The instantaneous price impact typically combines depth and resilience of the market into one stylized quantity. Time-dependent liquidity in this setting then leads to executing the constant liquidity strategy in volume time or liquidity time, and no qualitatively new features occur. In a second group of models resilience is finite and depth and resilience are separately modelled, e.g., [6], [11], [1] and [12]. Our model falls into this last group. Allowing for time-dependent depth and resilience leads to higher technical complexity, but allows us to capture a wider range of real world phenomena.

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Arrow-Debreu equilibria for rank-dependent utilities

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We provide conditions on a pure exchange economy with rank-dependent utility agents under which Arrow-Debreu equilibria exist. When such an equilibrium exists, we derive the state-price density *explicitly*, which is a weighted marginal rate of substitution between initial and end-of-period consumption of a representative agent, while the weight is expressed through the differential of the probability weighting function. A key step in our derivation is to obtain an analytical solution to the individual consumption optimization problem that involves the concave envelope of certain non-concave function. Based on the result we have several findings, including that asset prices depend upon agents' subjective belief on overall consumption growth, that an uncorrelated security's entire probability distribution and its dependence with the other part of the economy should be priced, and that there is a direction of thinking about the equity premium puzzle and the risk-free rate puzzle. Moreover, we propose a "rank-neutral probability" that is an appropriate modification of the original probability measure under which assets can be priced in the same way as in an economy with expected utility agents.

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Contributed talks

Stochastic mechanical systems with unilateral state constraints: control prospects

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1. There are two important intensively investigated fields in the theory of mechanical systems: systems with unilateral state constraints [1] and systems with random perturbations [2]. We begin a pioneering work in the overlap of these fields.

2. Consider a system, describing a particle moving in the axis with specular reflection at $\{0\}$: $t \geq 0$,

$$\begin{aligned} dX_t = Y_t\,dt + d\psi_t, \qquad X_t \geq 0, \qquad X_0 = x \geq 0, \\ dY_t = b(X_t, Y_t)\,dt + dW_t + d\varphi_t, \qquad Y_{0-} = y; \end{aligned} \tag{1}$$

$$\begin{aligned} d\psi_t \geq 0, \\ \Delta\psi_t = 0, \qquad \psi_0 = 0; \end{aligned} \tag{2}$$

$$\begin{aligned} d\phi_t = d\phi_t I_{\{0\}}(X_t) \geq 0, \\ \Delta\phi_t = 2|Y_{t-}| I_{(-\infty,0]}(Y_{t-}), \qquad \phi_{0-} = 0. \end{aligned} \tag{3}$$

Here W is a random perturbation — the Wiener process — and ϕ is a random nondecreasing process, describing the reflection impacts on the velocity. The process ψ has no physical interpretation. Probably it equals zero, but it is not proven. At present we know only that $d\psi_t = d\psi_t I_{\{0,0\}}(X_t, Y_t)$.

The problem (1), (2) does not fit into the pattern of reflected diffusions in sense of Skorokhod.

Our ultimate aim is to transfer the results of [3] to the object (1)-(3). [3] consideres a control problem for a process in a convex domain, reflected in sense of Skorokhod. The payoff functional has infinite horizon and time discount. The problems of existence and optimization for the process are solved (Theorem 2.2 [3]) by a penalization method, which gives great opportunities for numerical calculations. The specular reflection operator is far and away more difficult to work with than the Skorokhod one. At present we do not know how to apply the penalization, and our achievement is the proof of existence of a solution of (1)-(3) by another method.

Theorem 1. *Assume that b is continuous and of no more than linear growth. Then for any initial conditions the solution exists (a weak one, in the sense of distributions).*

The proof uses the approach for SDE with reflecting boundaries in sense of Skorokhod: ϕ is approximated with a special discretization, and the correspondent

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solutions converge weakly as random processes. In the next two paragraphs we describe "cornerstones" of the proof.

3. Basing on (1)-(3), we can define a specular reflection operator acting on $f \in C_b^1$. Let us construct its ε-approximation. Hitting level $-\varepsilon$ at moment t, X^ε gets a jump $\Delta X_t^\varepsilon = \varepsilon$, and Y^ε — $\Delta Y_t^\varepsilon = 2|\Delta Y_{t-}^\varepsilon|$.

Lemma 2. *Let $t > 0$, $\Delta\varphi_t^\varepsilon > 0$. Then $f'(t) < 0$, $\varphi_{t-}^\varepsilon < |f'(t)|$, and $\varphi_t^\varepsilon < 3|f'(t)|$.*

Proof. $\Delta\varphi_t^\varepsilon > 0$ implies $Y_{t-}^\varepsilon < 0$, so $f'(t) + \varphi_{t-}^\varepsilon < 0$, which yields: $f'(t) < 0$; $|Y_{t-}^\varepsilon| + \varphi_{t-}^\varepsilon = |f'(t)|$ and $|Y_{t-}^\varepsilon| \le |f'(t)|$. Finally, $\Delta\varphi_t^\varepsilon = 2|Y_{t-}^\varepsilon| \le 2|f'(t)|$. □

Now take $\delta > 0$ and construct a partition: $t_0 = 0, t_1 =$ the first moment X^ε hits $-\varepsilon$, ..., $t_{i+1} =$ the first moment X^ε hits $-\varepsilon$ after $t_i + \delta$,... This partition assures: modulus of continuity of φ is "majorized" by that of f'. Indeed, let us shift the problem to $[t_i, \infty)$ and the function

$$X_{t_i}^\varepsilon + \int_{t_i}^t (\varphi_{t_i} + f'(s))ds.$$

The behavior of Y^ε on $[t_i, t_{i+1})$ may be regarded as an ε-approximation of $(f' + \varphi_{t_i})$. Applying then Lemma 2 we get $\varphi_{t_{i+1}} - \varphi_{t_i} \le -(f' + \varphi_{t_i})(t_i)$. Since $f' + \varphi_{t_{i+1}} \ge 0$, the right hand side is majorized by

$$\operatorname*{osc}_{[t_i, t_{i+1})} (f' + \varphi_{t_i}) = \operatorname*{osc}_{[t_i, t_{i+1})} f'.$$

4. Operator ψ^ε is an approximation of the Skorokhod reflection for function $X^\varepsilon - \psi^\varepsilon$. The difference of this function and f is non-decreasing. This allows to "majorize" ψ^ε by the Skorokhod reflection for f (which is equal to $-(\min_{s\in[0,t)} f(s) \wedge 0)$); see Theorem 1 [4]. And if the value of the solution of (1)-(3) at moment t ($X_t = 0, Y_{t-} < 0$), then the jump $\Delta\varphi_t$ is isolated (on the time axis). But as a continuous function ψ can't grow at an isolated point.

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Threshold strategies in optimal stopping and free-boundary problems

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1. Let $X_t,\ t \geq 0$ be a homogeneous diffusion process defined on a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{P})$ with values in an interval $\mathcal{I} =]l, r[\subset \mathbb{R}^1,\ -\infty \leq l < r \leq \infty$ and the infinitesimal operator $\mathbb{L}_X f(x) = a(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x)$.

We assume that X_t is a regular process with starting point $X_0 = x$, and left endpoint l of the state space is either a natural or an entry-not-exit point.

Let us consider the following optimal stopping problem for this process:

$$V(x) = \sup_{\tau \in \mathfrak{M}} \mathbf{E}^x g(X_\tau) e^{-\rho\tau}, \tag{1}$$

where $g : \mathcal{I} \to \mathbb{R}^1$ is a reward function (continuous and bounded below), $\rho > 0$ is discount rate, and maximum is taken over some class $\mathfrak{M}$ of stopping times (s.t.) τ (with respect to 'natural' filtration $\mathcal{F}_t^X = \sigma\{X_s,\ 0 \leq s \leq t\},\ t \geq 0$).

2. Let us consider the class of stopping times induced by 'threshold strategies', i.e. the class $\mathfrak{M}_0$ of $\tau_p = \inf\{t \geq 0 : X_t \geq p\}$ (first time when process X_t exits the interval $]l, p[$) for all $p \in \mathcal{I}$.

The optimal stopping problem (1) over the class $\mathfrak{M}_0$ can be rewritten as follows:

$$V^*(x) = \sup_{p \in \mathcal{I}} \mathbf{E}^x g(X_{\tau_p}) e^{-\rho\tau_p}. \tag{2}$$

We will call a set $\{V^*(x) > g(x)\}$ as a continuation set for the problem (2).

Define the function $h(p) = g(p)/\psi(p),\quad p \in \mathcal{I}$, where $\psi(x)$ is the unique (up to a multiplicator) increasing solution to the equation $\mathbb{L}_X u(x) = \rho u(x)$ at the interval $\mathcal{I}$.

The following result gives the necessary and sufficient conditions for a continuation set in the problem (2) will be an interval.

Theorem 1. *The interval $]l, p^*[$, where $l < p^* < r$, is the continuation set for the problem (2) if and only if the following conditions hold:*

$$h(p) < h(p^*) \text{ whenever } p < p^*; \quad h(p) \text{ does not increase for } p > p^*. \tag{3}$$

In particular, a boundary of a continuation set p^* is a maximum point of the function h. This implies the necessity (under minimal assumptions) of smooth-pasting principle (see also [1, 2]).

The conditions (3) are remained necessary also for the optimal stopping problem (1) over *all* stopping times $\mathfrak{M}$ if the continuation set $\{V(x) > g(x)\}$ is an interval

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$]l, p^*]$. Under some additional assumptions these conditions will be sufficient also for the continuation set in the problem (1) over *all* stopping times $\mathfrak{M}$.

Theorem 2. *Let for some $p^* \in \mathcal{I}$ the conditions (3) hold and, moreover, $g \in C^2(]p^*, r[)$; $\tau_{p^*} = \inf\{t \geq 0 : X_t \geq p^*\} < \infty$ (a.s.) for $x < p^*$; $\mathbb{L}_X g(x) \leq \rho g(x)$ for $x > p^*$. Then $\{V(x) > g(x)\} =]l, p^*[$, i.e. the continuation set in the problem (1) over all stopping times is the interval $]l, p^*[$.*

3. For the considered threshold strategies the free-boundary problem can be written as follows: to find a threshold $\bar{p}$ and a function $U(x)$, $l < x < \bar{p}$, such that

$$\mathbb{L}_X U(x) = \rho U(x), \quad l < x < \bar{p}; \tag{4}$$

$$U(\bar{p} - 0) = g(\bar{p}); \tag{5}$$

$$U'(\bar{p} - 0) = g'(\bar{p}). \tag{6}$$

The conditions (4)–(5) hold for the function $U(x) = h(\bar{p})\psi(x)$ (for $x < \bar{p}$), and smooth-pasting condition (6) is equivalent to stationarity h at the point $\bar{p}$, i.e. $h'(\bar{p}) = 0$. There are simple examples for which the free-boundary problem has several solutions, or does not give a solution to optimal stopping problem (see, e.g., [3]).

Let us denote $V_p(x) = \mathbf{E}^x g(X_{\tau_p}) e^{-\rho\tau_p}$ for any $x, p \in \mathcal{I}$.

Theorem 3. *Let $(U(x), p^*)$ be a solution to free-boundary problem (4)-(6) and $g \in C^2(\mathbb{R}^1)$. Then:*

1) if $U''(p^*) > g''(p^*)$, then p^* is the point of local maximum (in p) of the function $V_p(x)$, moreover, for $x < p^*$ we have $V_p(x) < V_{p^*}(x)$ at some neighbourhood of p^*;

2) if $U''(p^*) < g''(p^*)$, then p^* is the point of local minimum (in p) of the function $V_p(x)$, moreover, for $x < p^*$ we have $V_p(x) > V_{p^*}(x)$ at some neighbourhood of p^*.

The case $U''(p^*) = g''(p^*)$ needs the additional considerations and maybe application of high-order conditions.

For the case of several solutions to free-boundary problem Theorem 3 allows to discard such solutions that do not solve the optimal stopping problem (e.g. giving local minimum in p to function $V_p(x)$).

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Predicting the ultimate maximum of a Lévy process

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Optimal prediction of the ultimate maximum is a non-standard optimal stopping problem in the sense that the pay-off function depends on a process which is not adapted to the given filtration, in this case the ultimate maximum. For a finite time horizon, this problem has been studied in various papers including Graversen, S. E. and Peskir, G. and Shiryaev, A. N. (2001 Theory Probab. Appl.), Du Toit, J. and Peskir, G. (2009 Ann. Appl. Probab.), Bernyk, V., Dalang, R. C. and Peskir, G. (2011 Ann. Probab.). In this work we consider the infinite horizon case for a Lévy process drifting to minus infinity. We also find a more explicit expression for the optimal stopping time in the spectrally one-sided case.

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Optimal production management when demand depends on the business cycle

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We assume that consumer demand for an item follows a Brownian motion with drift that is modulated by a continuous-time Markov chain that represents the regime of the economy. The economy may be in either one of two regimes, it remains in one regime for a random amount of time that is exponentially distributed with rate λ_1, and then moves to the other regime and remains there for an exponentially distributed amount of time with rate λ_2. Management of the company would like to maintain the inventory level of the item as close as possible to a target inventory level and would also like to produce the items at a rate that is as close as possible to a target production rate. The company is penalized by the deviations from the target levels and the objective is to minimize the total discounted penalty costs over the long term. We consider two models. In the first model the management of the company observes the regime of the economy at all times, whereas in the second model the management does not observe the regime of the economy. We solve both problems and obtain the optimal production policy as well as the minimal total expected discounted cost. Our analytical results show, among various other results, that in both models the optimal production policy depends on factors that are based on short term concerns as well as factors that are based on long term concerns. We analyze how the impact of these factors depend on the values of the parameters in the model. In addition, we compare the total expected discounted costs of the two models with one another and determine the value of knowing the current regime of the economy. We also solve the above problems when the cumulative consumer demand follows a geometric Brownian motion that is modulated by the continuous-time Markov chain that represents the regime of the economy.

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On estimate for variational inequality associated to optimal stopping

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1. We consider a probability space $(\Omega, \mathcal{F}, P)$ and an n-dimensional Wiener process $w_t = (w_t^1, \dots, w_t^n)$ on it. Let D be a bounded domain in $\mathbb{R}^n$ with a smooth boundary ($\partial D \in C^2$). Denote $\sigma(D) = \inf\{t \geq 0 : \ w_t \overline{\in} D\}$, and let $g = g(x)$, $c = c(x)$ be continuous functions defined on $\overline{D}$. Denote also by P_x the probability measure corresponding to the initial condition $w_0(\omega) = x$ and define the following optimal stopping problem

$$S(x) = \sup_{\tau \in \mathfrak{M}} E_x \left(g(w_\tau) I_{(\tau < \sigma(D))} + \int\limits_0^{\tau \wedge \sigma(D)} c(w_s)\, ds \right), \tag{1}$$

where $\mathfrak{M}$ is the class of all stopping times with respect to the filtration $F^w = (\mathcal{F}_t^w)_{t \geq 0}$. The optimal stopping problem consists in finding a payoff $S(x)$ and in defining the optimal stopping time τ^* at with the supremum (1) is achieved [1].

2. Denote by $H^1(D)$ the first order Sobolev space of functions $v = v(x)$ on D and let $H_0^1(D)$ be the subspace of $H^1(D)$ consisting of the functions $v = v(x)$, "equal to zero" on the boundary ∂D. Denote $K = \{v : \ v \in H_0^1(D), \ v(x) \geq g(x)\}$ and let $a(u, v)$ be scalar product in $H_0^1(D)$. The variational inequality is formulated as follows: find a function $u(x) \in K$ such that the inequality

$$a(u, v - u) \geq \int\limits_D c(x)(v(x) - u(x))\, dx \tag{2}$$

is fulfilled for any function $v(x) \in K$. In [2] A. Bensoussan has established the fundamental connection between the optimal stopping problem and the corresponding variational inequality. In particular, he has shown that

$$u(x) = S(x), \ \ x \in \overline{D}, \tag{3}$$

and the estimate

$$\sup_{x \in \overline{D}} |u^2(x) - u^1(x)| \leq \sup_{x \in \overline{D}} |g^2(x) - g^1(x)| \tag{4}$$

holds, where the functions $u^i(x)$, $i = 1, 2$, represent the solutions of the variational inequality (2) for the functions $g^i(x)$, $i = 1, 2$.

3. Via the stochastic analysis the present paper gives an answer to the following question: does the uniform closeness of the functions $g^1(s)$ and $g^2(x)$ imply in a

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certain sense the closeness of the partial derivatives $\frac{\partial u^1(x)}{\partial x_i}$, $\frac{\partial u^2(x)}{\partial x_i}$, $i = 1, \dots, n$, of the corresponding solutions $u^1(x)$ and $u^2(x)$ of the variational inequality (2)? Using the results from [1] and [3] , we obtain a new estimate which is formulated as follows.
Let $g^i(x)$, $c^i(x)$, $i = 1, 2$, be two initial pairs of the variational inequality (2). Than for the solutions $u^i(x)$, $i = 1, 2$, of the problem (2) the global estimate

$$\int_D d^2(x, \partial D)\big|\mathrm{grad}\,(u^2 - u^1)(x)\big|^2\,dx + \int_D (u^2(x) - u^1(x))^2\,dx \le$$
$$\le C\Big[\Big(\sup_{x\in D}|g^2(x) - g^1(x)| + \sup_{x\in D}|c^2(x) - c^1(x)|\Big)\times$$
$$\times\Big(\sup_{x\in D}|g^1(x)| + \sup_{x\in D}|c^1(x)| + \sup_{x\in D}|g^2(x)| + \sup_{x\in D}|c^2(x)|\Big)\Big]$$

is valid, where $d(x, \partial D)$ is the distance from the point x to the boundary ∂D, C is a constant depending on the dimension of the space $\mathbb{R}^n$ and on the Lebesgue measure of D, i.e. $C = C(n, mes(D))$.
In [4] analogous estimates are used for optimal portfolio in the pricing problem of American type options.

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Optimal investment with random innovations

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We consider a simplified model of a firm whose performance is a function of the technology level. The firm operates with an initial technology level m and the current (best) technology available is assumed to be a renewal process N_t with discrete increments. At any moment the firm can switch to the best technology available, incurring an investment cost. We seek the strategy (i.e., the best investment time) maximizing the life-time discounted value of the firm.

We provide a general characterization of the optimal solution. For some particular structures of the renewal process' intensity, it is possible to derive explicit solutions.

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On a structure of a minimax test in testing composite hypotheses

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1. We consider the problem of testing two composite hypotheses in the minimax setting. Namely, let $\mathscr{P}$ ("null hypothesis") and $\mathscr{Q}$ ("alternative hypothesis") be two families of probability measures on a measurable space $(\Omega, \mathscr{F})$. It is assumed that both families are dominated, say, by a probability measure R; E stands for expectation with respect to R. It is convenient for us to identify measures from $\mathscr{P}$ and $\mathscr{Q}$ with their densities with respect to R. We denote by Φ the set of all randomized tests, i.e., measurable functions $\varphi\colon \Omega \to [0,1]$. The set Φ_α, $\alpha \in [0,1]$, of tests of level α is defined by

$$\Phi_\alpha := \bigl\{\varphi \in \Phi\colon \mathsf{E}[p\varphi] \le \alpha \text{ for every } p \in \mathscr{P}\bigr\}.$$

The problem is to find a minimax test, i.e. a test $\varphi^* \in \Phi_\alpha$ such that

$$\inf_{q\in\mathscr{Q}} \mathsf{E}[q\varphi^*] = \sup_{\varphi\in\Phi_\alpha} \inf_{q\in\mathscr{Q}} \mathsf{E}[q\varphi] =: v(\alpha).$$

It is well known that a minimax test always exists.

2. Put $\alpha_0 := \sup\limits_{\mathsf{P}\in\mathscr{P}} \inf\limits_{A\in\mathscr{F}\colon \mathsf{Q}(A)=1\ \forall \mathsf{Q}\in\mathscr{Q}} \mathsf{P}(A)$, $\beta_0 := \sup\limits_{\mathsf{Q}\in\mathscr{Q}} \inf\limits_{A\in\mathscr{F}\colon \mathsf{P}(A)=1\ \forall \mathsf{P}\in\mathscr{P}} \mathsf{Q}(A)$. It is easy to check that $v(0) = 1-\beta_0$ and $v(\alpha) = 1$ iff $\alpha \ge \alpha_0$ and to find minimax tests for such values of α.

In what follows $\mathrm{co}(\cdot)$ stands for the convex hull and bar means the closure with respect to the convergence in R-probability. Define a functional $F(p,q,z)$, $p \in \overline{\mathrm{co}(\mathscr{P})}$, $q \in \overline{\mathrm{co}(\mathscr{Q})}$, $z > 0$, by

$$F(p,q,z) := -\mathsf{E}[q \wedge (zp)] + \alpha z + 1. \tag{1}$$

Introduce the following (dual) minimization problem:

$$F(p,q,z) \longrightarrow \min; \quad p \in \overline{\mathrm{co}(\mathscr{P})},\ q \in \overline{\mathrm{co}(\mathscr{Q})},\ z > 0. \tag{2}$$

Theorem 1. *Let $0 < \alpha < \alpha_0$.*

(i) *If $\varphi^* \in \Phi_\alpha$ and $(p^*, q^*, z^*) \in \overline{\mathrm{co}(\mathscr{P})} \times \overline{\mathrm{co}(\mathscr{Q})} \times (0,\infty)$ are such that*

$$\varphi^* = \begin{cases} 1, & \text{if } q^* > z^* p^*, \\ 0, & \text{if } q^* < z^* p^*, \end{cases} \tag{3}$$

$$\mathsf{E}[p^*\varphi^*] = \alpha, \tag{4}$$

$$\mathsf{E}[q^*(\varphi^*-1)] = \inf_{q\in\mathscr{Q}} \mathsf{E}[q(\varphi^*-1)], \tag{5}$$

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then φ^* *is a minimax test and* (p^*, q^*, z^*) *is a solution to* (2).

(ii) *Assume that the family* $\{p \wedge q\colon p \in \mathrm{co}(\mathscr{P}),\, q \in \mathrm{co}(\mathscr{Q})\}$ *is* R*-uniformly integrable. Then* (2) *has, at least, one solution,*

$$\min_{(p,q,z)\in\overline{\mathrm{co}(\mathscr{P})}\times\overline{\mathrm{co}(\mathscr{Q})}\times(0,\infty)} F(p,q,z) = v(\alpha),$$

and (3)–(5) *hold for any minimax test* φ^* *and any solution* (p^*, q^*, z^*) *to* (2).

Remark 1. This result is symmetrical between $\mathscr{P}$ and $\mathscr{Q}$. In particular, $\alpha \rightsquigarrow 1 - v(\alpha)$ is a decreasing one-to-one mapping from $(0, \alpha_0)$ onto $(0, \beta_0)$, φ^* is a minimax test of level $\alpha \in (0, \alpha_0)$ for testing $\mathscr{P}$ against $\mathscr{Q}$ if and only if $1 - \varphi^*$ is a minimax test of level $1 - v(\alpha)$ for testing $\mathscr{Q}$ against $\mathscr{P}$, and (p^*, q^*, z^*) is a solution to the dual problem (2) if and only if $(q^*, p^*, 1/z^*)$ is a solution to the problem obtained from (2) after exchanging $\mathscr{P}$ and $\mathscr{Q}$ and replacing α by $1 - v(\alpha)$.

3. The sufficiency of conditions (3)–(5) for $\varphi^* \in \Phi_\alpha$ to be a minimax test is a classical result established in Lehmann [3], where p^* and q^* are assumed to be Bayesian mixtures of measures from $\mathscr{P}$ and $\mathscr{Q}$. It is easy to show that these sets of mixtures, $\mathrm{mix}(\mathscr{P})$ and $\mathrm{mix}(\mathscr{Q})$, are subsets of $\overline{\mathrm{co}(\mathscr{P})}$ and $\overline{\mathrm{co}(\mathscr{Q})}$, respectively. A method to find optimal Bayesian mixtures was suggested by Krafft and Witting [2] who introduced a dual problem similar to (2), p and q being varied over $\mathrm{mix}(\mathscr{P})$ and $\mathrm{mix}(\mathscr{Q})$. However, a solution of the minimization problem over such a domain exists under rather strong assumptions. To ensure the existence of a solution, Cvitanić and Karatzas [1] considered a dual problem with not necessarily probability densities. Namely, they assumed that $q \in \mathscr{Q} = \overline{\mathrm{co}(\mathscr{Q})}$ (which implies the uniform integrability of $\mathscr{Q}$) and $p \in \{p \geq 0\colon \mathsf{E}[p\varphi] \leq \alpha$ for every $\varphi \in \Phi_\alpha\}$ and proved a result similar to ours. However, they did not observe that the above set can be replaced by a smaller set $\overline{\mathrm{co}(\mathscr{P})}$. This last remark allows us to prove directly the main result in Rudloff and Karatzas [4] as well. Finally, let us mention that in all the papers mentioned above the variable q is always a probability density. In such a case, the functional F in (1) and relation (5) can be written alternatively as

$$F(p,q,z) = \mathsf{E}[(q - zp)^+] + \alpha z, \qquad \mathsf{E}[q^*\varphi^*] = \inf_{q\in\mathscr{Q}} \mathsf{E}[q\varphi^*],$$

as is done in these papers.

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Net gain problem with two stops for an urn scheme

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In this paper, the following optimal double stopping problem on trajectories is considered. Suppose that there is an urn containing m balls of value -1 and p balls of value $+1$. The player is allowed to draw ball randomly, without replacement, one by one. The value -1 is attached to minus ball and value $+1$ to plus ball. Determine sequence $Z_0 = 0$, $Z_n = \sum_{k=1}^{n} X_k$, $1 \le n \le m+p$, where X_k is the value of the ball chosen at the k-th draw. The player observes the values of the balls and wants to make two stops. The aim of player is to maximize the expected gain, the gain is difference between maximum and minimum values of the trajectory formed by $\{Z_n\}_{n=0}^{m+p}$ (net gain problem).

This urn scheme could be considered as the buying-selling problem. Here the value of the ball is change of the cost of an asset. The first stop means the buying of an asset and the second stop is the selling of an asset. In net gain problem the player wants to maximize the difference between costs.

The urn schemes with one stop was considered by Shepp L. (1969) (net gain problem), Tamaki M. (2001) (max-problem), Mazalov V.V., Tamaki M. (2007) (duration problem).

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On a two-side disorder problem for a Brownian motion in a Bayesian setting

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1. Suppose we sequentially observe a stochastic process $X = (X_t)_{t\geq 0}$ having the structure

$$dX_t = \mu \mathbf{I}(t \geq \theta)dt + dB_t,$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, $\theta > 0$ and μ are *unobservable* random variables with known distributions, independent mutually and of B. The random variable θ is the moment when the drift of X_t changes its value from zero to μ, i.e. *"disorder"* happens.

In this paper we consider the case when random variables θ and μ have the following structure: θ takes value 0 with probability p ($q = 1 - p$ below) and it is exponentially distributed with parameter $\lambda > 0$ given that $\theta > 0$; μ takes values $\mu_1 < 0$ and $\mu_2 > 0$ with corresponding probabilities ρ_1 and $\rho_2 = 1 - \rho_1$. Being based upon the continuous observation of X our task is to detect the moment of disorder θ and define the value of μ (to test μ for hypotheses $H_1 : \mu = \mu_1$ and $H_2 : \mu = \mu_2$) with minimal loss.

For this, we consider *a sequential decision rule* $\delta = (\tau, d)$, where τ is a stopping time of the observed process X (with respect to the natural filtration $(\mathcal{F}_t^X)_{t\geq 0}$), and d is an $\mathcal{F}_\tau^X$-measurable random variable taking values d_1 and d_2. After stopping the observation at time τ the terminal decision d indicates which hypothesis on the drift value should be accepted: if $d = d_1$ we accept H_1 and if $d = d_2$ we accept H_2.

With each decision rule $\delta = (\tau, d)$ we associate the *Bayesian risk*

$$\mathbb{R}(\delta) = \mathbb{R}^\theta(\delta) + \mathbb{R}^\mu(\delta),$$

where

$$\mathbb{R}^\theta(\delta) = \mathsf{P}(\tau < \theta) + c\mathsf{E}[\tau - \theta]^+$$

is a combination of the probability of a "false alarm" and the average delay in detecting the "disorder" correctly, $c > 0$ is a given constant, and

$$\mathbb{R}^\mu(\delta) = a\mathsf{P}(d = d_1, \mu = \mu_2) + b\mathsf{P}(d = d_2, \mu = \mu_1)$$

is the average loss due to a wrong terminal decision, where $a > 0$ and $b > 0$ are given constants.

The problem then consists of finding the decision rule $\delta_* = (\tau_*, d_*)$ such that

$$\mathbb{R}(\delta_*) = \inf_\delta \mathbb{R}(\delta), \tag{1}$$

where the infimum is taken over all decision rules δ.

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Thus, the problem under consideration combines the classical problems of detecting the "disorder" and sequential hypothesis testing (for details see e.g. [1], Chapter VI).

2. Introduce the a posteriori probability processes $\pi^i = (\pi^i_t)_{t\geq 0}$, $i = 1, 2$ with

$$\pi^i_t = \mathsf{P}(\theta \leq t, \mu = \mu_i \,|\, \mathcal{F}^X_t), \quad i = 1, 2.$$

The method of solution of (1) is natural in such kind of problems and consists in reduction to an optimal stopping problem.

Theorem 1. *The 2-dimensional process $\pi = (\pi^1, \pi^2)$ is a Markov sufficient statistic in problem* (1)*. Moreover, the process π solves the following system of stochastic differential equations:*

$$d\pi^i_t = \lambda\rho_i(1 - \pi^1_t - \pi^2_t)dt + \pi^i_t \left[\frac{\mu_i}{\sigma} - \left(\frac{\mu_1}{\sigma}\pi^1_t + \frac{\mu_2}{\sigma}\pi^2_t\right)\right] d\overline{B}_t, \quad i = 1, 2,$$

where $\overline{B} = (\overline{B}_t)_{t\geq 0}$ is a Brownian motion (generally, different from B_t). The optimal stopping time τ_* can be found as the solution of the optimal stopping problem

$$V(\pi) = \inf_\tau \mathsf{E}_\pi \Bigg[1 - \pi^1_\tau - \pi^2_\tau + c\int_0^\tau (\pi^1_t + \pi^2_t)\, dt \\ + a(\rho_1\pi^2_\tau + \rho_2(1 - \pi^1_\tau)) \wedge b(\rho_2\pi^1_\tau + \rho_1(1 - \pi^2_\tau))\Bigg], \quad (2)$$

where E_π denotes the mathematical expectation with respect to the measure P_π, under which π_t starts P_π-a.s. from the point π. Terminal decision function is defined as $d_* = d_1$ if $a(\rho_1\pi^2_\tau + \rho_2(1 - \pi^1_\tau)) < b(\rho_2\pi^1_\tau + \rho_1(1 - \pi^2_\tau))$ and $d_* = d_2$ otherwise.

In the talk we discuss analytical properties of the optimal stopping rules in the problem (2) and show how to compute optimal stopping boundary numerically.

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Symmetric integrals and stochastic analysis

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1. In this paper following [1] we consider a symmetric integral $\int_0^t f(s, X(s)) * dX(s)$ with respect to an arbitrary continuous function $X(s)$. If $X(s)$ is a path of Brownian motion, then the symmetric integral coincides with the Stratonovich integral.

Let $0 = t_0^{(n)} < t_1^{(n)} < ... < t_{N_n}^{(n)} = t$ be a sequence of partitions such that $\lim_{n\to\infty} \max_k (t_k^{(n)} - t_{k-1}^{(n)}) \to 0$. The limit $\lim\limits_{n\to\infty} \int_0^t f(s, X^{(n)}(s))(X^{(n)})'(s)ds$ is called a symmetric integral and is denoted by $\int_0^t f(s, X(s)) * dX(s)$. Here $X^{(n)}(s)$ denotes a broken line.

Suppose that for almost all u:

(a) $f(s, u)$, $s \in [0, t]$, is a right-continuous bounded variation function;

(b) the total variation $|f|(t, u)$ of the function $f(s, u)$, $s \in [0, t]$, is an integrable function;

(c) $\int_0^t \mathbf{1}(s : X(s) = u)|f|(ds, u) = 0$;

then there exists a symmetric integral $\int_0^t f(s, X(s)) * dX(s)$.

The symmetric integral $\int_0^t f(s, X(s)) * dX(s)$ has the following properties:

(i) Let assumptions (a) – (c) hold, then

$$\int_0^t f(s, X(s)) * dX(s) = \int_{X(0)}^{X(t)} f(t, u)du - \int_R \int_0^t \kappa(u, X(0), X(s)) f(ds, u)du,$$

here $\kappa(u, a, b) = sign(b - a)\mathbf{1}(a \wedge b < v < a \vee b)$.

(ii) Suppose that $F(t, u)$ has continuous partial derivatives F'_t, F'_u; then

$$F(t, X(t)) - F(0, X(0)) = \int_0^t F_u(s, X(s)) * dX(s) + \int_0^t F_s(s, X(s))ds.$$

2. A scalar first-order pathwise differential equation in differential form is written as the following equation

$$d\xi_s = \sigma(s, X(s), \xi_s) * dX(s) + b(s, X(s), \xi_s)ds, \quad \xi_0 = \xi(0), \quad s \in [0, t_0]. \qquad (1)$$

Here the first term in the right-hand corresponds to a symmetric integral, and the second term corresponds to a Riemann integral. The function $\xi(s) = \phi(s, X(s))$ is called a solution if the following conditions hold:

(i) the function $\phi(s, v)$ has continuous partial derivatives $\varphi'_v(s, v)$, $\varphi''_{sv}(s, v)$;

(ii) the function $\xi(s) = \phi(s, X(s))$ satisfies (1).

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From now on we make the assumption: the continuous function $X(s)$ is almost nowhere differentiable. The existence of solution of pathwise equation can be guaranteed by the following theorem.

Theorem 1 *Suppose that the functions $\sigma(s,v,\phi)$, $\sigma'_s(s,v,\phi)$, $\sigma'_\phi(s,v,\phi)$, $b(s,v,\phi)$ jointly continuous; then the following conditions are equivalent:*

(i) there exist a solution $\xi(s)=\phi(s,X(s))$;

(ii) the function $\xi(s)=\phi(s,u)$, $\varphi(0,X(0))=\xi(0)$, for almost all s satisfies the condition
$\phi'_v(s,X(s))=\sigma(s,X(s),\phi(s,X(s)))$; $\phi'_s(s,X(s))=b(s,X(s),\phi(s,X(s)))$.

Theorem 2 *Let all assumptions of Theorem 1 hold. Suppose that the function $b'_\phi(s,v,\phi)$ is jointly continuous; then there exists a unique solution of equation (1).*

Remark 1 Let $\sigma(s,v,\phi)\neq 0$. Using Theorem 1, we obtain the following equations chain

$$\phi'_v(s,v)=\sigma(s,v,\phi);\quad \phi'_s(s,X(s))=b(s,X(s),\phi(s,X(s))).$$

To find a solution of (1), we need to find a solution of this chain of equations.

For example, suppose that $\xi_t-\xi_0=\int_0^t[a\xi_s+b]*dX(s)+\int_0^t[h\xi_s+g]ds$ is a linear pathwise equation with respect to the symmetric integral. From Remark 1 it follows that $\phi'_u(t,u)=a\phi(t,u)+b$, $\phi'_t(t,u)|_{u=X(t)}=h\phi(t,X(t))+g$, $\phi(0,X(0))=\xi_0$. Hence $\phi(t,u)=\frac{1}{a}\left(e^{u+C(t)}-b\right)$, where $C(s)$ is an arbitrary function. In order to find a function $C(s)$, it is necessary to solve the equation $\frac{1}{a}e^{X(t)+C(t)}C'(t)=\frac{h}{a}\left(e^{X(t)+C(t)}-b\right)+g$ with initial condition $\frac{1}{a}\left(e^{X(0)+C(0))}-b\right)=\xi_0$.

3. The results of section 2 can be extended to more complex equations.

(i) Consider the equation $\eta(t)-\eta(0)=\sum\limits_{k=1}^{d}\int\limits_0^t a_k(s,\eta(s))*dW_k(s)+\int\limits_0^t b(s,\eta(s))ds$, $t\in[0,T]$, where $(W_1(s),...,W_d(s))$ is a multi-dimensional Brownian motion. The solution of this equation must be sought in the form of $\eta(s)=\phi(s,W_1(s),...,W_d(s))$. To find $\eta(s)$, it is necessary to solve the equations chain

$$\phi'_{u_k}(s,W_1(s),...,W_{k-1}(s),u_k,W_{k+1}(s),...,W_d(s))=$$

$$=a_k(s,\phi(s,W_1(s),...,W_{k-1}(s),u_k,W_{k+1}(s),...,W_d(s)),\ k=1,...,d,$$

$$\phi'_s(s,W_1(s),...,W_d(s))=b(s,\phi'_s(s,W_1(s),...,W_d(s))).$$

(ii) Similarly, for the evolutional differential equation

$$u(t,x)-u(0,x)=\int_0^t F_1\left(s,x,X(s),u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)ds+$$

$$+\int_0^t F_2\left(s,x,X(s),u,\frac{\partial u}{\partial x_1},...,\frac{\partial^k u}{\partial x_1^{k_1}...\partial x_n^{k_n}},...\right)*dX(s),\quad (s,x)\in R^+\times R^n,$$

$k_1 + ... + k_n = k \leq m$, the solution is sought in the form of $u(s,x) = u(s,x,X(s))$. To find the solution of this equation, it is necessary to solve the equations chain

$$\frac{\partial}{\partial v} u(s,x,v) = F_2 \left(s, x, v, u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial^k u}{\partial x_1^{k_1} ... \partial x_n^{k_n}}, ... \right) \Big|_{u=u(s,x,v)},$$

$$\frac{\partial}{\partial s} u(s,x,v)|_{v=X(s)} = F_1 \left(s, x, X(s), u, \frac{\partial u}{\partial x_1}, ..., \frac{\partial^k u}{\partial x_1^{k_1} ... \partial x_n^{k_n}}, ... \right) \Big|_{u=u(s,x,X(s))}.$$

Note that this method can be applied to solve the problem of nonlinear filtering of diffusion processes.

4. The linearization problem (see [1] for more details) of the stochastic ordinary differential equations is to find a change of variables such that a transformed equation becomes a linear equation.

Theorem 3 *Suppose that the coefficients σ and b of the equation (1) are continuous and $\sigma \neq 0$. Then (1) is reducible to the linear differential equation $d\eta_t = A(t)\eta_t * dX(t) + B(t)\eta_t dt$.*

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Locally most powerful group-sequential tests when the groups are formed randomly

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1. Let a sequential statistical experiment be conducted in the following way: at each stage, we observe a group of random variables $(X_{i,1}, X_{i,2}, \ldots, X_{i,n_i})$; the observations $X_{i,j}$ are i.i.d. with the distribution P_θ, $\theta \in \Theta$, where Θ is an open subset of the real line, and the size of the i-th group, n_i, is defined by a random variable ν_i. Let κ_n, a non-decreasing function of n, be the cost to obtain a group of n observations (in the simplest case $\kappa_n \equiv 1$). We consider a problem of testing a simple hypothesis $H_0 : \theta = \theta_0$ against a composite alternative $H_1 : \theta > \theta_0$, where $\theta_0 \in \Theta$ is some fixed point.

The goal is to construct a locally most powerful (LMP) test for this problem, i.e. a test maximizing the slope of the power function at $\theta = \theta_0$ in the class of all sequential tests such that type I error and the average overall cost of the experiment do not exceed the given constants.

2. Let f_θ be the probability density function (a Radon-Nikodym derivative) of P_θ, with respect to some σ-finite measure μ, for all $\theta \in \Theta$.

Suppose the fulfillment of the following conditions:

C1. $\exists \gamma_1 : \limsup_{\theta \to \theta_0} I(\theta_0, \theta)/(\theta - \theta_0)^2 = \gamma_1 < \infty$,
where $I(\theta_0, \theta) = E_{\theta_0} \ln f_{\theta_0}(X_1)/f_\theta(X_1)$ is the Kullback-Leibler information.

C2. $\exists\ \dot{f}_{\theta_0}$: $\dot{f}_{\theta_0}$ is integrable (with respect to μ) and

$$\int |f_\theta - f_{\theta_0} - (\theta - \theta_0)\dot{f}_{\theta_0}| d\mu = o(\theta - \theta_0),$$

as $\theta \to \theta_0$, i.e. $\dot{f}_{\theta_0}$ is the Fréchet derivative of f_θ at $\theta = \theta_0$ in $L_1(\mu)$.

C3. $\exists \delta_1 < \infty : E\nu_i < \delta_1\ \forall i \in \mathbb{N}$.

C4. $\exists \delta_2, \delta_3 : 0 < \delta_2 \le E\kappa_{\nu_i} \le \delta_3 < \infty\ \forall i \in \mathbb{N}$.

3. Then the LMP group-sequential test is defined by the stopping time

$$\tau = \inf \left\{ k : \sum_{i=1}^{k} \sum_{j=1}^{n_i} \frac{\dot{f}_{\theta_0}(X_{i,j})}{f_{\theta_0}(X_{i,j})} \notin (-A, B) \right\}$$

and the terminal decision to reject H_0 if

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} \dot{f}_{\theta_0}(X_{i,j})/f_{\theta_0}(X_{i,j}) > B,$$

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where A and B are some positive numbers.

4. In case $P(\nu_i = 1) = 1$ for all $i \geq 1$, the LMP group-sequential test becomes a LMP sequential test considered in [1], [2], [3]. An optimal group-sequential test for testing a simple hypothsis against a simple alternative for discrete distributions was considered in [4].

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Optimal stopping of geometric Brownian motion with partial reflection

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We consider a problem of optimal stopping for geometric Brownian motion Z_t on $]0,\infty[$ with parameters b,σ and killing intensity r. at the point x there is a partial reflection, so that $P_x[Z_t > x] \to (1+\alpha)/2$ as $t \to 0$, $-1 < \alpha < 1$. The payoff function $\bar{g}(z) = (z-K)^+$. The value function is $V(z) = \sup\limits_{\tau} E_z \bar{g}(Z_\tau)$, where supremum is taken over all stopping times τ.

An algorithm of constructing the value function for the problem of optimal stopping of one-dimensional regular diffusion with finite number of singular points was given in [1]. The term "singular point" refers to points of partial reflection of diffusion, the points of discontinuities of payoff function and its first derivative, as well as the ends of intervals, where application of the first revaluation operator to the payoff function takes positive values. The algorithm is based on the notion of modification of the payoff function (modification does not change the value function).

Preliminary results were given in [2]. In particular the case of Brownian motion with partial reflection was considered. After reading the paper [2] M.Zervos decided to consider the case of geometric Brownian motion with partial reflection and gave a talk on The Sixth Bachelier Colloquium: Mathematical Finance and Stochastic Calculus, January 15-22, 2012, Métabief, France [3].

A dependence of the optimal strategy on the values of three parameters (x,α,K) was described in [3] for the case $b < r$. The proof was based on the variational inequalities. We show that without restriction of generality one can consider only the case $K = 1$ and give the transparent picture of the dependence (see Fig. 1). We consider also the case $b = r$. In case $b > r$ one has $V(z) \equiv \infty$.

Our approach is based only on the references to the results of [1] and explicit calculation of $f_{]c,\,d[}(z) = E_z f(Z_{\tau_{]c,\,d[}})$ where $\tau_{]c,\,d[} = \inf\{t : \ t \geq 0, \quad Z_t \notin]c,\,d[\}$. If $f_{]c,\,d[}(z) > f(z)$ for $z \in]c,\,d[$ then according to [1] the function $f_{]c,\,d[}(z)$ is a modification of $f(z)$. The function $f_{]c,\,d[}(z)$ satisfies on $]c,\,d[$ the equations $Lf(z) = 0$, $L_1 f(z) = 0$, where the revaluation operators L and L_1 for this problem are

$$Lf(z) := \left(\sigma^2 z^2/2\right) f''(z) + bzf'(z) - rf(z) \text{ for } z \neq x,$$

$$L_1 f(z) := (1+\alpha(z))f'_+(z) - (1-\alpha(z))f'_-(z), \text{ where } \alpha(z) = \begin{cases} \alpha & \text{for } z = x, \\ 0 & \text{for } z \neq x. \end{cases}$$

The general solution of the equation $Lf(z) = 0$ has a form $az^n + Bz^m$, where n and $m < n$ are the solutions of the equation $\dfrac{\sigma^2}{2}\lambda(\lambda-1) + b\lambda - r = 0$.

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The function $\bar{g}(z)$ in case $K = 1$ is a modification of $g(z) = z - 1$. Therefore instead of $\bar{g}(z)$ it is more convenient to consider the problem with the payoff function $g(z)$. Since $Lg(z) = (b - r)z - r$ there are two singular points: x and $x_b = \dfrac{r}{r-b}$. An important role plays also the point $x_n = \dfrac{n}{n-1}$ such that $]x_n, \infty[$ is a stopping set for the case without partial reflection ($\alpha = 0$). It follows from the equation for m and n that $n > 1$, $m < 0$, and that $r/b > n$, so that $x_b < x_n$.

It follows from [1] that $g^{(1)}(z) = g_{]0,\min(x,x_b)[}(z)$ is a modification of $g(z)$ and, if $x_b > x$, then also $g^{(2)}(z) = g^{(1)}_{]x,x_b[}(z)$ is a modification of $g(z)$. Further modifications in the neighborhoods of points x and x_b give the following result.

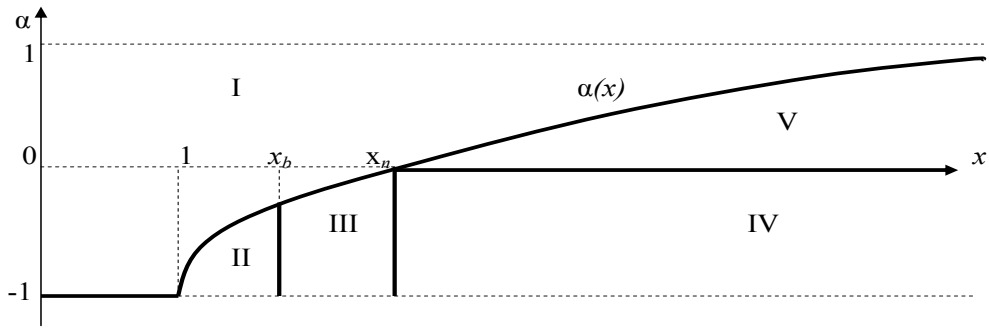

Figure 1: Partition of $B = \{(x, \alpha) : x > 0, \alpha \in]-1, 1[\}$ in accordance with the character of the stopping set D:
I. If $\alpha > \alpha(x)$ then $D = [d(x, \alpha), \infty[$, where $d(x, \alpha) > x$;
II. If $1 < x < x_b$ and $-1 < \alpha \le \alpha(x)$, then $D = \{x\} \bigcup [d(x), \infty[$, where $d(x) > x$;
III. If $x_b \le x \le x_n$ and $-1 < \alpha \le \alpha(x)$ then $D = [x, \infty[$;
IV. If $x_n < x < \infty$ and $-1 < \alpha \le 0$ then $D = [x_n, \infty[$;
V. If $x_n < x < \infty$ and $0 < \alpha \le \alpha(x)$ then $D = [x_n, c(x, \alpha)[\bigcup [d(x, \alpha), \infty[$ where $c(x, \alpha) < x < d(x, \alpha)$ for $\alpha < \alpha(x)$ and $c(x, \alpha(x)) = x = d(x, \alpha(x))$.
The values $d(x, \alpha), d(x)$ and $c(x, \alpha)$ are defined from the smooth-fitting conditions.

If $r = b, K = 1$ then there are two cases. If $x > 1, \alpha < \alpha^*$ for some $\alpha^* > 0$ then for $z < x$ the optimal stopping time is the time of the first visit of x, and for $z > x$ one has $V(z) = \lim_{d\to\infty} g_{]x,d[}(z)$. Otherwise $V(z) = \lim_{d\to\infty} g_{]0,d[}(z)$.

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An optimal dividend and investment control problem under debt constraints

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In this work, we consider the problem of determining the optimal control on the dividend and investment policy of a firm. There are a number of research on this corporate finance problem. In [1], Décamps and Villeneuve study the interactions between dividend policy and irreversible investment decision in a growth opportunity and under uncertainty. We may equally refer to [2] for an extension of this study, where the authors relax the irreversible feature of the growth opportunity. In other words, they consider a firm with a technology in place that has the opportunity to invest in a new technology that increases its profitability. The firm self-finances the opportunity cost on its cash reserve. Once installed, the manager can decide to return back to the old technology by receiving some cash compensation.

As in a large part of the literature in corporate finance, the above papers assume that the firm cash reserve follows a drifted Brownian motion. They also assume that the firm does not have the ability to raise any debt for its investment as it holds no debt in its balance sheet. In our study, as in the Merton model, we consider that firm value follows a geometric Brownian process and more importantly we consider that the firm carries a debt obligation in its balance sheet. However, as in most studies, we still assume that the firm assets is highly liquid and may be assimilated to cash equivalents or cash reserve. We allow the company to make investment and finance it through debt issuance/raising, which would impact its capital structure and risk profile. This debt financing results therefore in higher interest rate on the firms outstanding debts. Furthermore, we consider that the manager of the firm works in the interest of the shareholders, but only to a certain extent. Indeed, in the objective function, we introduce a penalty cost P and assume that the manager does not completely try to maximize the shareholders value since it applies a penalty cost in the case of bankruptcy. This penalty cost could represent, for instance, an estimated cost of the negative image upon his/her own reputation due to the bankruptcy under his management leadership. Mathematically, we formulate this problem as a combined singular and multiple-regime switching control problem. Each regime corresponds to a level of debt obligation held by the firm.

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Stochastic control and free boundary problem for sailboat trajectory optimization

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I will present a stochastic control problem motivated by sailing races. The goal is to minimize the travel time between two locations, by selecting the fastest route in face of randomly changing weather conditions, such as wind direction. When a sailboat is travelling upwind, the key is to decide when to tack, that is, to switch bearings in such a way that if the wind was coming from one side of the yacht before the tacking, then, after the tacking, it comes from the other side of the yacht. Since this maneuver slows down the yacht, it is natural to model this time lost by a *tacking penalty*, $c > 0$. This places the problem in the context of optimal stochastic control problems with switching costs.

Our model preserves some of the real-world features of wind variability while eliminating some of the geometric problems arising from the specifics of yacht motion. We assume that the *yacht's speed* v is a constant function of the angle γ between the yacht's bearing and the wind direction: $v(\gamma) = v\,\mathbf{1}_{\{|\gamma|\geq\frac{\pi}{4}\}}$, $\gamma \in [-\pi, \pi]$. This is a big simplification which still preserves the main features of the initial problem. Indeed, though the speed of the yacht certainly depends on γ, assuming that sailing settings are chosen so as to maximize the yacht's upwind velocity, the yacht sails mostly at a nearly constant speed. Furthermore, in order to simplify the problem, we consider that the wind speed is also constant and that only the *wind direction* $(W_t)_{t\geq 0}$ is random. We assume that (W_t) is a two-state continuous-time Markov chain defined on a filtered space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, taking values in $\{\pm\alpha\}$ with $\alpha \in]0, \frac{\pi}{4}[$. A *tacking strategy* is defined as a right continuous, piecewise constant and adapted process $(A_t)_{t\geq 0}$ taking values in $\{\pm 1\}$. If $A_t = 1$ (resp. -1), it means that at time t, the yacht is sailing on starboard (resp. port) tack. Namely, the wind enters the sails from its right (resp. left) side. The *number of tackings* of a strategy A is given by the following process:

$$N_t(A) = \sharp\{s \in [0, t] : A_s \neq A_{s-}\}.$$

To ensure model consistency, a strategy is *admissible* only if it satisfies some extra conditions which will be given during the presentation. If one starts at a point $\vec{x}$ of the race area, on a tack $a \in \{\pm 1\}$ and under a wind $w \in \{\pm\alpha\}$, then the *payoff function* of a race driven by an admissible tacking strategy A is given by

$$J(\vec{x}, a, w, A) = \mathbb{E}_{\vec{x},a,w}\left(\tau^A + cN_{\tau^A}(A)\right)$$

where τ^A is the hitting time of the target buoy. The *value function* of the problem is then given by

$$V(\vec{x}, a, w) = \inf_{A \text{ admiss.}} J(\vec{x}, a, w, A).$$

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First, we will discuss some properties of the solution and the concept of a *lifted tack*, specific to route optimization. This allows us to characterize the wind into two categories (stable or unstable) related to the mean time between changes in wind direction. Several asymptotic cases have been studied in [2]. Here, I would like to present a particular case where it is possible to find an explicit solution of the problem. We assume that the yacht starts close to the target buoy under a stable wind. In this case, we can show that the value function solves a system of first order partial differential equations with free boundaries that are easy to find. The system can be transformed into second order hyperbolic partial differential equations of Klein-Gordon type. We compute explicitly the solutions of these equations, which give formulas for the value function of the problem. A verification theorem establishes then the optimality of the solution. I will conclude by giving the general shape of the solution when we consider the problem in the entire state space and the procedure to compute the value function in that case.

This work has been done in collaboration with R. Dalang and was highly motivated by the work of F. Dumas in [1].

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A general Bayesian disorder problem for a Brownian motion on a finite interval

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1. Suppose we sequentially observe a process $X = (X_t)_{t\geq 0}$ satisfying the equation

$$dX_t = \mu \mathbf{I}(t \geq \theta)dt + dB_t, \qquad X_0 = 0,$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion, θ is a non-negative *unobservable* random variable taking values in an interval $[0, T]$ with a known distribution, independent of B, and $\mu > 0$ is a known number. The random variable θ can be interpreted as the moment of disorder – the moment when the drift of X changes. We consider the problem of detecting the disorder that consists in finding the stopping time τ^* of the filtration $(\mathcal{F}_t^X)_{t\geq 0}$ which is "as close as possible" to θ.

Let $H(t)$ be *a penalty function*, which decreases for $t < 0$, increases for $t > 0$ and $H(0) = 0$. Mathematically, we look for the stopping time τ^* such that

$$\mathsf{E}H(\tau^* - \theta) = \inf_{\tau} \mathsf{E}H(\tau - \theta), \tag{1}$$

where the infimum is taken over all stopping times τ of $(\mathcal{F}_t^X)_{t\geq 0}$.

Some particular cases of problem (1) have been considered in the literature mainly when θ is exponentially distributed and $H(t)$ is of a special form (see the review in [1]). In the present paper we provide a general solution to the problem when θ takes values in a finite interval $[0, T]$.

2. We consider the case when $H(t)$ is linear or exponential for $t \geq 0$, i. e.

$$H(t) = ct \ \text{ for } t \geq 0 \qquad \text{or} \qquad H(t) = \frac{c}{b}e^{bt} \ \text{ for } t \geq 0,$$

where $b, c > 0$ are known numbers. In the linear case, for convenience, we assume $b = 0$. Introduce *the generalized Shiryaev–Roberts statistic* $\psi = (\psi_t^{(b)})_{t\geq 0}$:

$$\psi_t^{(b)} = e^{\mu X_t - (\mu^2/2 - b)t} \int_0^t e^{-\mu X_s + (\mu^2/2-b)s} dG(s).$$

Let E^∞ denote the mathematical expectation with respect to the measure, under which X is a Brownian motion, and define function $\tilde{H}(t) = \int_t^\infty H(t-s)dG(s)$.

We show that under mild smoothness conditions on G, the optimal stopping time τ^* in problem (1) can be found as the first hitting time of $\psi^{(b)}$ to a time-dependent level:

$$\tau^* = \inf\{t \geq 0 : \psi_t^{(b)} \geq a(t)\} \wedge T,$$

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where the function $a\colon [0,T] \to \mathbb{R}_+$ is the unique continuous solution of the equation

$$\int_t^T \mathsf{E}^\infty\big[(c\psi_s^{(b)} + \tilde{H}'(s))\mathbf{I}\{\psi_s^{(b)} < a(s)\} \,\big|\, \psi_t^{(b)} = a(t)\big]ds = 0, \quad t \in [0,T],$$

satisfying the conditions

$$a(t) \geq -\tilde{H}'(t)/c \text{ for } t < T, \qquad a(T) = -\tilde{H}'(T-)/c.$$

The average penalty $\mathcal{H} = \mathsf{E}H(\tau^* - \theta)$ can be found by the formula

$$\mathcal{H} = \tilde{H}(0) - \int_0^T \mathsf{E}^\infty\big[(c\psi_s^{(b)} + \tilde{H}'(s))\mathbf{I}\{\psi_s^{(b)} < a(s)\}\big]ds.$$

3. The main idea consists in reducing problem (1) to the following optimal stopping problem for the process $\psi^{(b)}$:

$$V = \inf_{\tau \leq T} \mathsf{E}^\infty \left[\tilde{H}(\tau) + c\int_0^\tau \psi_s^{(b)} ds\right].$$

The solution of this problem is found using standard methods.

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Posters

Efficient hedging of options with robust convex loss functionals

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1. We study the problem of partial hedging a European option in an incomplete financial market, modeled through a semimartingale discounted price process $S = (S_t)_{t\in[0,T]}$ on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathsf{P})$, $T < \infty$.

Let P_σ denote the set of probability measures P^* equivalent to P such that S is a sigma-martingale with respect to P^*. We assume that S satisfies the condition of NFLVR (no free lunch with vanishing risk). As in [1], it is equivalent to $P_\sigma \neq \varnothing$.

We model the discounted payoff of a European option with an $\mathcal{F}_T$-measurable, nonnegative random variable H and assume $C_0 := \sup_{\mathsf{P}^*\in P_\sigma} \mathsf{E}^{\mathsf{P}^*}[H] < \infty$.

Then for a given initial capital $0 \le c \le C_0$ we have the following optimization problem

$$u(c) = \inf_{\xi\in Adm} \sup_{\mathsf{Q}\in\mathcal{Q}} \big\{\mathsf{E}^{\mathsf{Q}}[l(H - V_T^\xi)^+] - \gamma(\mathsf{Q})\big\}, \tag{1}$$

where $\mathcal{Q}$ is a convex family of absolutely continuous probability measures with respect to P, $l\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a convex, nondecreasing loss function, $\gamma\colon \mathcal{Q} \to \mathbb{R}_+$ is a convex function, Adm is a class of all admissible hedging strategies and $V_t^\xi := c + \int_0^t \xi_s dS_s$ is the corresponding value process, $t \in [0, T]$.

2. Efficient hedging was introduced and solved in a general semimartingale model in continuous time in [4]. The authors used the expected loss function as a risk measure.

The two-steps method of [4] can be analogously applied to our case and provides that

$$u(c) = \inf_{V\in\mathbb{A}} \sup_{\mathsf{Q}\in\mathcal{Q}} \big\{\mathsf{E}^{\mathsf{Q}}[l(H - V)] - \gamma(\mathsf{Q})\big\}, \tag{2}$$

where $\mathbb{A} := \big\{V \in \mathcal{F}_T \mid \mathsf{P}(0 \le V \le H) = 1 \text{ and } \sup_{\mathsf{P}^*\in P_\sigma} \mathsf{E}^{\mathsf{P}^*}[V] \le c\big\}$.

3. In the present paper we provide a dual characterization of the value function of this optimal problem.

To be more concrete, the following dual-representation formula holds:

$$u(c) = -\inf_{y\ge 0} \inf_{\eta\in\mathcal{D},\mathsf{Q}\in\mathcal{Q}} \big\{\mathsf{E}^{\mathsf{Q}}[\tilde{V}_H(y\frac{\eta}{Z^{\mathsf{Q}}})] + \gamma(\mathsf{Q}) + \mathsf{c}y\big\}, \tag{3}$$

where $Z^{\mathsf{Q}} = d\mathsf{Q}/d\mathsf{P}$, $\quad \tilde{V}_{H(w)}(\lambda) = \sup_{0\le x\le H(w)}\big\{-\lambda x - l(H(w) - x)\big\}$ and $\mathcal{D} := \big\{\eta \in L_1^+ \mid \mathsf{E}^{\mathsf{P}}\eta V \le 1 \text{ for all } V \in \mathbb{A}\big\}$.

Moreover the infimum in (2) is attained.

4. The same result is obtained in [5], but the authors used additional assumptions to prove it. In this paper it is shown that one can prove some facts by using

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the convex analysis methods and do not require additional assumptions such as conditions on differentiability of the loss function in [5].

In particular, we widely use a new approach to the notion of the f-divergence [2,3] which extends the domain of its definition to bounded finitely additive set functions taking nonnegative values.

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Expected utility maximization in exponential Lévy models

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1. In modern financial mathematics the problem of maximizing expected utility of the asset portfolio has become increasingly popular.

In our work we consider a model of a financial market with one asset and a finite maturity T. The capital process has the form $X = x + H \cdot S$, where x is the initial wealth, H is a predictable process (also called "strategy") and S is a semimartingale that models the asset's price. We assume that all capital processes belong to the set $\mathcal{X}(x) = \{X_t \geq 0 : X_0 = x\}$. Our aim is to maximize the expected logarithmic utility at time T:

$$u(x) = \sup_{X \in \mathcal{X}(x)} E[\ln(X_T)].$$

Here we suppose that $u(x) < +\infty$.

The problem of utility maximization was considered in [1] by Kramkov and Schachermayer in a general model of incomplete markets and a general utility function finite on $\mathbb{R}_+$. The solution was found by solving the dual problem, where the minimum was taken over the set of supermartingale deflators, not only equivalent local martingale measures.

In our work we consider the case of exponential Lévy models when S is the stochastic exponential of a Lévy process L, $\Delta L \geq -1$, with the triplet (b, c, ν). The problem of maximizing logarithmic utility in exponential Lévy models was solved by Hurd [2] under the assumption that the logarithms of price processes have jumps unbounded from above and below. He used the dual method and indicated that there are cases where the solution of the dual problem is a supermartingale and not necessary a martingale.

It is well known that the solution X^* is the numéraire portfolio and the solution of the dual problem satisfies $X^*Y^* = 1$. Kardaras [3] showed that the numéraire portfolio exists in an exponential Lévy model iff the process L is not monotonous. Recall [4] that the Lévy process is monotonous iff $c = 0,\ \nu[x < 0] = 0,\ b - \int x\mathbb{1}_{|x|\leq 1}\nu(dx) \geq 0$ or $c = 0,\ \nu[x > 0] = 0,\ b - \int x\mathbb{1}_{|x|\leq 1}\nu(dx) \leq 0$. We show that if the monotonous assumption is not satisfied, the numéraire portfolio X^* exists and there are only three possibilities for $Y^* = 1/X^*$.

1. Y^* is a supermartingale, but not a martingale.
2. Y^* is a martingale, but not the density process of an equivalent σ-martingale measure.
3. Y^* is the density process of an equivalent martingale measure.

The aim of our work is to classify all these cases in terms of the Lévy triplet.

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2. Consider the set $\mathfrak{C} = \{p : \nu\{x : (1+px) < 0\} = 0\}$ and put $\overline{M} = \inf(\mathfrak{C})$, $\overline{N} = \sup(\mathfrak{C})$. Denote by L_1 and L_2 the following quantities:

$$L_1 = c\overline{N} - b + \int_{|x|\le 1} \frac{x^2}{|(1/\overline{N}) + x|}\nu(dx) - \int_{x>1} \frac{x}{1+\overline{N}x}\nu(dx)$$

$$L_2 = c\overline{M} - b + \int_{|x|\le 1} \frac{-x^2}{|(1/\overline{M}) + x|}\nu(dx) - \int_{x>1} \frac{x}{1+\overline{M}x}\nu(dx)$$

Here we use the rules: $0 \cdot \infty = 0,\ 1/\infty = 0,\ 1/0 = \infty$.

Theorem 1. *In a finite-time exponential Lévy model, for $Y^* = 1/X^*$, where X^* is the numéraire portfolio, the following holds true:*

1. *Y^* is the density process of an equivalent martingale measure in one of the following 3 cases:*

   (i) $$b + \int_{x>1} x\nu(dx) > 0 \text{ or } +\infty \text{ and}$$
   $$L_1 \ge 0 \text{ if } \overline{N} < +\infty \text{ or } L_1 > 0 \text{ if } \overline{N} = +\infty.$$

   (ii) $$b + \int_{x>1} x\nu(dx) = 0.$$

   (iii) $$b + \int_{x>1} x\nu(dx) < 0 \text{ and}$$
   $$L_2 \le 0 \text{ if } \overline{M} > -\infty \text{ or } L_2 < 0 \text{ if } \overline{M} = -\infty.$$

2. *Y^* is a martingale but not the density process of an equivalent martingale measure, when the following is satisfied:*
   $$b + \int_{x>1} x\nu(dx) < 0 \text{ and } \overline{M} = 0.$$

3. *Y^* is a supermartingale and not a martingale in one of the following 2 cases:*

   (i) $$b + \int_{x>1} x\nu(dx) > 0 \text{ or } +\infty,$$
   $$L_1 < 0 \text{ and } \overline{N} < +\infty.$$

   (ii) $$b + \int_{x>1} x\nu(dx) < 0,$$
   $$L_2 > 0 \text{ and } -\infty < \overline{M} < 0.$$

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On superhedging prices of contingent claims

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1. We consider a model of security market which consists of $d+1$ assets, one bond and d stocks. We suppose that the price of the bond is constant and denote by $S = (S^i)_{1\le i\le d}$ the price process of the d stocks. The process S is assumed to be a semimartingale on a given filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\le t\le T}, \mathsf{P})$. We assume that $T > 0$ is a finite time horizon and $\mathscr{F}_0$ is trivial, $\mathscr{F}_T = \mathscr{F}$.

Consider an investor on our financial market. A (self-financing) portfolio Π of the investor is a pair (x, H), where the constant x is the initial value of the portfolio and $H = (H^i)_{1\le i\le d}$ is a trading strategy of the investor, i.e. is a predictable S-integrable process specifying the amount of each asset held in the portfolio. The value process $X = (X_t)_{0\le t\le T}$ of such a portfolio Π is given by

$$X_t = X_0 + \int_0^t H_u dS_u, \quad 0 \le t \le T. \tag{1}$$

We denote by $\mathscr{H}$ the set of admissible trading strategies of the investor and by $\mathscr{X}(x)$ the family of wealth processes with non-negative capital at any instant, i.e. X is of the form (1), $X_t \ge 0$ for all $t \in [0, T]$, and with initial value equal to x.

2. Given a contingent claim B with maturity T, we consider the following two values:

$$\mathscr{V}_{\mathscr{H}}(B) = \inf\{x \in \mathbb{R} : \exists H \in \mathscr{H} : x + (H \cdot S)_T \ge B\} \tag{2}$$

and

$$\mathscr{V}_+(B) = \inf\{x \in \mathbb{R} : \exists X \in \mathscr{X}(x) : X_T \ge B\}. \tag{3}$$

The values $V_{\mathscr{H}}(B)$ and $\mathscr{V}_+(B)$ are called the superhedging prices of the claim B and are the smallest initial endowments that allow the investor to super-replicate B at maturity. But in the first case, investor is allowed to use trading strategies from the set $\mathscr{H}$, and in the second case, the wealth process of the investor has to be non-negative.

3. Superhedging was introduced and investigated first by El Karoui and Quenez [1] in a continuous-time setting where the risky assets follow a multidimensional diffusion process. Delbaen and Schachermayer [2, 3] generalized these results to, respectively, a locally bounded and unbounded semimartingale model, under the $(NFLVR)$ condition. Theorem 1 extends the results of papers [2, 3]. In Theorem 2 we prove a new representation of the price $\mathscr{V}_+(B)$ via the sets $\mathscr{Z}^s$ and $\mathscr{Z}^\sigma$ of supermartingale and σ-martingale densities respectively (see [4]).

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4. Let us firstly introduce some basic objects we need to formulate our main results, Theorems 1 and 2. Denote by $\mathscr{A}$ the following set: $\mathscr{A} = \{(H \cdot S)_T,\ H \in \mathscr{H}\}$. Let $\psi = 1 + |B|$. Then we construct the sets $\mathscr{C}^\psi$ and $\mathscr{R}$ by the following rules:

$$\mathscr{C}^\psi = (\mathscr{A} - L^0_+) \cap (\psi L^\infty),$$

$$\mathscr{R} = \left\{ \mu \in ba_+ : \mu\left(\frac{1}{\psi}\right) = 1,\ \mu(\xi) \leqslant 0\ \forall \xi \in \frac{\mathscr{C}^\psi}{\psi} \right\}.$$

The elements of $\mathscr{R}$ are usually called separating measures in the literature, analogous to the concept of martingale measure, but in a more general setting.

Theorem 1. Assume that the set $\mathscr{H}$ is a convex cone, $\mathscr{V}_{\mathscr{H}}(\psi) < \infty$ and $\frac{B}{\psi} \in L^\infty$. Then

$$\mathscr{V}_{\mathscr{H}}(B) = \max_{\mu \in \mathscr{R}} \mu\left(\frac{B}{\psi}\right). \tag{4}$$

If $B \in L^\infty$ and $\mathscr{H}$ is a set $\mathscr{H}^{bb}$ of bounded from below wealth processes, then, under the $(NFLVR)$ condition, formula (4) is reduced to the result of paper [2]. We can also reduce our formula to the one from paper [3], but under the additional assumption $\mathscr{V}_{\mathscr{H}^\psi}(\psi) < \infty$, where $\mathscr{H}^\psi$ is a set of ψ-admissible trading strategies, introduced in [3]. In comparison with papers [2, 3], we use an abstract class of trading strategies, also we do not need any assumptions on arbitrage on financial market and the maximum in formula (4) is attained.

Theorem 2. Assume that $B \in L^0_+$. Then, under the $(NUPBR)$ condition (see [4]),

$$\mathscr{V}_+(B) = \sup_{Z \in \mathscr{Z}^s} \mathsf{E}\, BZ_T = \sup_{Z \in \mathscr{Z}^\sigma} \mathsf{E}\, BZ_T. \tag{5}$$

If $\mathscr{H} = \mathscr{H}^{bb}$, then, in general, $\mathscr{V}_{\mathscr{H}}(B) \leq \mathscr{V}_+(B)$. It is easy to prove that, under the $(NFLVR)$ condition, $\mathscr{V}_{\mathscr{H}}(B) = \mathscr{V}_+(B)$. We give an example which shows that, under the weaker $(NUPBR)$ condition, it is possible to have $-\infty < \mathscr{V}_{\mathscr{H}}(B) < \mathscr{V}_+(B) < +\infty$.

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Sharp inequalities for maximum of skew Brownian motion

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Let $W^\alpha = (W_t^\alpha)_{t\geq 0}$ be a skew Brownian motion with parameter $\alpha \in [0,1]$ which can be defined as a unique strong solution $X = (X_t)_{t\geq 0}$ of stochastic equation

$$X_t = X_0 + B_t + (2\alpha - 1)L_t^0(X),$$

where $L_t^0(X)$ is the local time at zero of X_t. In the present work we obtain maximal inequalities for skew Brownian motion. These inequalities generalize well-known results concerning standard Brownian motion $B = (B_t)_{t\geq 0}$ (case $\alpha = 1/2$) and its modulus $|B| = (|B_t|)_{t\geq 0}$ (case $\alpha = 1$). Namely, the authors of [1], [2] established that for any Markov time $\tau \in \mathfrak{M}$

$$\mathsf{E}(\max_{0\leq t\leq \tau} B_t) \leq \sqrt{\mathsf{E}\tau}, \quad \mathsf{E}(\max_{0\leq t\leq \tau} |B_t|) \leq \sqrt{2\mathsf{E}\tau},$$

where $\mathfrak{M}$ is the set of all Markov times τ (with respect to the natural filtration of B) with $\mathsf{E}\tau < \infty$. The main result of our work is contained in the following theorem (see [3]).

Theorem 1. *For any Markov time $\tau \in \mathfrak{M}$ and for any $\alpha \in (0,1)$ we have*

$$\mathsf{E}\left(\max_{0\leq t\leq \tau} W_t^\alpha\right) \leq M_\alpha \sqrt{\mathsf{E}\tau}, \tag{1}$$

where $M_\alpha = \alpha(1 + A_\alpha)/(1-\alpha)$ and A_α is the unique solution of the equation

$$A_\alpha e^{A_\alpha + 1} = \frac{1-2\alpha}{\alpha^2},$$

such that $A_\alpha > -1$. Inequality (1) is "sharp," i.e. for each $T \geq 0$ there exists a stopping time τ such that $\mathsf{E}\tau = T$ and

$$\mathsf{E}\left(\max_{0\leq t\leq \tau} W_t^\alpha\right) = M_\alpha \sqrt{\mathsf{E}\tau}.$$

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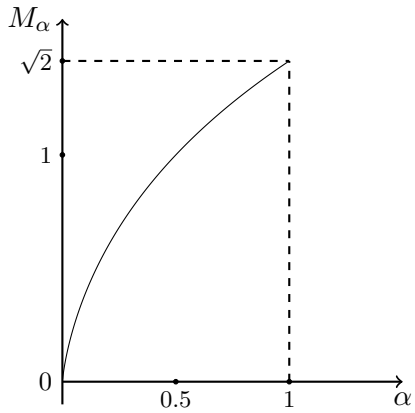

Fig. 1. The quantity M_α

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The stochastic optimization of the automated forecast of the severe squalls and tornadoes on the base of hydrodynamic-statistical forecast models

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Development of successful method of automated statistical well-in-advance forecast (from 12 hours to two days) of dangerous phenomena – severe squalls and tornadoes – could allow to mitigate the losses. The prediction of these phenomena is a very difficult problem for the synoptic till recently. The synoptic forecast of these phenomena is usual the subjective decision of an operator. Nowadays there is no successful hydrodynamic model for the forecast of such phenomena with the wind velocity more 20 m/s and more 24 m/s, hence the main tools for the objective forecast development are the methods using the statistical model of these phenomena recognition.

The meteorological situation involved the dangerous phenomena – the squalls and tornadoes with the wind velocity $V \geq 20$ m/s is submitted as the vector $\mathbf{X}(A) = (x_1(), x_2(), \ldots, x_n())$, where n – the quantity of the empiric potential atmospheric parameters (predictors). The values of these predictors for the dates and towns, where are these phenomena, were accumulated in the set $\{\mathbf{X}(A)\}$ – the learned sample of the phenomena A presence. The learned sample of the phenomena A absence or the phenomena B presence ($\{\mathbf{X}(B)\}$) was obtained for such towns, where the atmosphere was instability and the thunderstorms and the rainfalls were observed, but the wind velocity is not so high ($V <$ 8-10 m/s). The recognition model of the sets $\{\mathbf{X}(A)\}$ and $\{\mathbf{X}(B)\}$ was constructed with the help of Byes approach [1, 3]. This approach allow to minimize the middle economic losses of forecast errors (of the I and II kinds).

It was necessary to decide the problem of the compressing the predictors space without the information losses in order to choose the informative vector-predictor and to calculate the decisive rules of the recognition of the sets $\{\mathbf{X}(A)\}$ and $\{\mathbf{X}(B)\}$. It was made with the help of diagonalization of a sample matrix **R**algorithm [3]. The most informative predictors – representatives from each of diagonal blocks and two independent predictors are composed vector-predictor of dimension $k = 6$ (from $n = 26$ potential predictors) [3]. The most informative were estimated using the criterion by Mahalanobis distance Δ^2 ($\Delta^2 = (m_i(A) - m_i(B)) \uparrow 2/\sigma^2$) and criterion of the entropy minimum H_{min} by Vapnik-Chervonenkis [2, 3].

As a result, the informative vector-predictor of the most informative and slightly dependent predictors from six atmospheric parameters after this selection ($\mathbf{V_{700}}$, $\mathbf{T_{ea}}$, $\mathbf{Td_{ea}}$, $\mathbf{H_0}$, $(\mathbf{T} - \mathbf{T'})_{\mathbf{500}}$, $\mathbf{dT/dn_{ea}}$) [3]. Here $\mathbf{V_{700}}$ – the value of the mean velocity of the wind on the level 700 hPa, $\mathbf{T_{ea}}$ – the maximal value of the temperature near the earth level, $\mathbf{Td_{ea}}$ – the maximal value of the dew point near the earth

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level, $\mathbf{H_0}$ – the level of the isotherm of 0^oC, $(\mathbf{T'} - \mathbf{T})_{\mathbf{500}}$ – the difference between the values of the stratification curve and the moist adiabatic on the level 500 hPa, $\mathbf{dT/dn_{ea}}$ – the maximal difference between temperatures over the front on the earth level near the forecast point. Then the linear discriminant function $U(\mathbf{X})$, depended from these parameters, was calculated by Byes approach. If the value of $U(\mathbf{X}) > 0$ at the fix station, we have the forecast squall ($V > 20$ m/s) near this station during the current day. The tornadoes objective forecast examples (in Ivanovo, in Penza, in Dubna) were calculated by this statistical model with the using the discriminant function $U(\mathbf{X})$ (the value $U(\mathbf{X})$ was more than 3) [5]. The new statistical model and new discriminant functions $F_1(\mathbf{X})$ (for the wind velocity $V > 20$ m/s) and $F_2(\mathbf{X})$ (for the wind velocity V¿24 m/s) were develop on the base of the output data of the first hydrodynamic hemispheric model. The probability of dangerous winds for each of two classes $P_1(\mathbf{X})$ and $P_2(\mathbf{X})$ ($P_1(\mathbf{X}) = 100/(1 + \exp(F_1(\mathbf{X}))$; $P_2(\mathbf{X}) = 100/(1+\exp(-F_2(\mathbf{X})))$ were calculated operative in the nodes of the greed 150x150km two times per day. The probability more than the empiric threshold $\mathbf{P}$ give us the forecast area of such squalls. We obtained by same way [3] the new informative vector-predictors for each classes ($k = 8$ from $n = 38$ new parameters). This forecast of dangerous squalls and tornadoes over European part of Russia was recommended in 1993 years for the using in synoptic practice [4, 5]. This method was also adapted for the territory of Siberia. The examples of last hydrodynamic-statistical forecast model of squalls and tornadoes using the new regional hydrodynamic model output data in the nodes of mesh 75x75 km are submitted at [5]. Three submitted stochastic models of automated forecast of dangerous squalls and tornadoes over the territory of Russia are used for the development of the stochastic optimization of forecasts with the minimum economical losses of forecast errors. The optimal stochastic decisive rule was composed by the empiric approach using three hydrodynamic-statistical models of the automated forecasts of squall and tornadoes to current and next days.

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Binary encoding in data series and early warning signals on financial crises

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1. Financial crises have strong impacts on the economy, including labor markets, household incomes, and the profitability of companies (see, e.g., [1]-[3]). It is therefore important to understand how financial crises develop and whether it is possible to detect early warning signals on financial crises.

Financial crises lie in the area of extreme events. Compared to fluctuations in values of the indicators of a system's performance, extreme events are usually understood as qualitative shifts in the system's behavior. In this context, signals on the upcoming extreme events can be characterized in terms of tendencies rather than predictions on particular quantities. Roughly, one can group the tendencies in two categories – tendencies to a crisis and tendencies to avoiding a crises. Under that paradigm, early warning signals can be treated in a binary way – as either "minus" signals registering a tendency to a crisis, or "plus" signals registering a tendency to avoiding a crisis (see [4]).

Based on this binary approach, we develop a three-stage research pattern for identifying tendencies to crises in application to two recent financial crises – the Dot-com crisis of 2001-2002, and the latest global financial crisis of 2008-2009.

2. A first stage is *recognition.* Assessing an eight-year-long financial time series (the Dow Jones Industrial Average and the Federal Reserve Interest Rate) preceding the Dot-com crisis of 2001-2002, we identify some "minus" and "plus" signals. We understand the "minus" signals as short (four-month-long) patterns in the time series, which occur, primarily, close to the time of the crisis, and the "plus" signals as those occurring, primarily, in earlier periods. We propose a *binary encoding rule* that transforms short data patterns into "minus" and "plus" signals.

A second stage is a *statistical analysis.* We use the binary encoding rule to transform a long (1954-2001) time series preceding the crisis of 2001-2002 into a sequence of "minus" and "plus" signals, and analyze the frequencies of a "minus" and a "plus" to follow each short *binary window* in the sequence (in our analysis each binary window is formed by three subsequent overlapping signals covering six months). We treat the frequencies as transition probabilities, which define a *binary random process* operating in the space of the binary windows. In our analysis the binary random process serves as a model describing the mechanism for the "plus" and "minus" signals to occur in the operation of the financial system under consideration. Two important features of the model are the following. Firstly, as ensured by the recognition analysis, the model recognizes early warning signals on

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the crisis of 2001-2002. Secondly, as ensured by the statistical analysis, the model captures the dynamics of signals occurring in a long historical time series.

A third stage is *testing the forecasting ability* of the model. We use the model to assess, retrospectively, the probability of a financial crisis to occur in October 2008 (the latest global financial crisis was registered in the period from October 2008 to mid-2009). We show that the probability grows steadily starting from October 2007 and reaches value 1 in August 2008.

We also find the probability of a crisis to occur in each month in the period November 2008 - August 2009. The behavior of the probability is similar to that found for October 2008, i.e., it starts to grow fast and reaches size 1 several months before the month, for which we calculate the probability. It is shown that the probability of a crisis to occur starting from September 2009 (in the period when the real economy showed signs of recovery) grows slightly but never reaches 1.

Thus, our binary stochastic model based on analysis of data preceding the crisis of 2001-2002, demonstrates an ability to register early warning signals on the global financial crisis of 2008-2009.

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