Asymptotically optimal discretization of hedging strategies with jumps

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1. A basic problem in mathematical finance is to replicate a random claim with $\mathcal{F}_T$-measurable payoff H_T with a portfolio involving only the underlying asset Y and cash. When Y follows a diffusion process of the form

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)dW_t, \tag{1}$$

it is known that under minimal assumptions, a random payoff depending only on the terminal value of the asset $H_T = H(Y_T)$ can be replicated with the so-called delta hedging strategy. However, to implement such a strategy, the hedging portfolio must be readjusted continuously, which is of course physically impossible and anyway irrelevant because of the presence of microstructure effects and transaction costs. For this reason, the optimal strategy is always replaced with a piecewise constant one, leading to a discretization error. The relevant question is then to find the optimal discretization dates. Of course, it is intuitively clear that readjusting the portfolio at regular deterministic intervals is not optimal. However, the optimal strategy for fixed n is very difficult to compute.

Fukasawa [1] simplifies this problem by assuming that the hedging portfolio is readjusted at high frequency. The performance of different discretization strategies can then be compared based on their asymptotic behavior as the number of readjustment dates n tends to infinity, rather than the performance for fixed n. Consider a discretization rule : a sequence of discretization strategies

$$0 = T_0^n < T_1^n < \cdots < T_j^n < \dots,$$

with $\sup_j |T_{j+1}^n - T_j^n| \to 0$ as $n \to \infty$ and let $N_T^n := \max\{j \geq 0; T_j^n \leq T\}$ be the total number of readjustment dates on the interval $[0, T]$ for given n. To compare different discretization rules in terms of their asymptotic behavior, Fukasawa [1] uses the criterion

$$\lim_{n\to\infty} E[N_T^n]E[\langle \mathcal{E}^n \rangle_T], \tag{2}$$

where $\langle \mathcal{E}^n \rangle$ is the quadratic variation of the semimartingale $(\mathcal{E}_t^n)_{t\geq 0}$. He finds that when the underlying asset is a continuous semimartingale, the functional (2) admits a nonzero lower bound over all discretization rules, and exhibits a specific explicit rule based on hitting times which attains this lower bound and is therefore called *asymptotically efficient*.

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While the above approach is quite natural and provides very explicit results, it fails to take into account important factors of market reality. First, the asymptotic functional (2) is somewhat ad hoc, and does not reflect any specific model for the transaction costs. Second, the continuity assumption, especially at relatively high frequencies, is not realistic.

The objective of our work is therefore two-fold. First, we develop a framework for characterizing the asymptotic efficiency of discretization strategies which takes into account the transaction costs. Second, we remove the continuity assumption in order to understand the effect of the activity of small jumps (often quantified by the Blumenthal-Getoor index) on the optimal discretization strategies.

2. Our goal is to study and compare different discretization rules for the stochastic integral

$$\int_0^T X_{t-} dY_t,$$

where X and Y are semimartingales with jumps. More precisely, our principal assumptions on the processes X and Y are

- The process Y is a $\mathbb{F}$-local martingale, whose predictable quadratic variation satisfies $\langle Y\rangle_t = \int_0^t A_s ds$, where the process (A_t) is càdlàg and locally bounded.
- The process X is a pure jump semimartingale defined via the stochastic representation

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \int_{|z|\leq 1} z(M-\mu)(ds\times dz) + \int_0^t \int_{|z|>1} zM(ds\times dz), \tag{3}$$

where M is the jump measure of X and μ is its predictable compensator, which satisfies additionally $\mu(dt\times dz) = dt \times \lambda_t K_t(z)\,\nu(dz)$, where λ is a positive càdlàg process, K is a random function which is in some sense "close to 1" when z is close to 0 and ν is a Lévy measure satisfying

$$x^\alpha \nu((x,\infty)) \to c_+ \quad \text{and} \quad x^\alpha \nu((-\infty,-x)) \to c_- \quad \text{when} \quad x\to 0. \tag{H_α}$$

for some $\alpha \in (1,2)$ and constants $c_+ \geq 0$ and $c_- \geq 0$ with $c_+ + c_- > 0$.

A *discretization rule* is a family of stopping times $(T_i^\varepsilon)_{i\geq 0}^{\varepsilon>0}$ parameterized by a nonnegative integer i and a positive real ε, such that for every $\varepsilon>0$, $0 = T_0^\varepsilon < T_1^\varepsilon < T_2^\varepsilon < \ldots$, and $\sup\{i : T_i^\varepsilon \leq T\} < \infty$. For a fixed discretization rule and a fixed ε, we denote $\eta(t) = \sup\{T_i^\varepsilon : T_i^\varepsilon \leq t\}$ and $N_T = \sup\{i : T_i^\varepsilon \leq T\}$.

The performance of a discretization rule is measured by the error functional $\mathcal{E}(\varepsilon) : (0,\infty) \to [0,\infty)$ and the cost functional $\mathcal{C}(\varepsilon) : (0,\infty)\to[0,\infty)$. We consider the error functional given by the L^2 error

$$\mathcal{E}(\varepsilon) := E\left[\left(\int_0^T (X_{t-} - X_{\eta(t)-})dY_t\right)^2\right] \tag{4}$$

and a family of cost functionals of the form

$$\mathcal{C}^\beta(\varepsilon) = E\left[\sum_{i\ge 1: T_i^\varepsilon \le T} |X_{T_i^\varepsilon} - X_{T_{i-1}^\varepsilon}|^\beta\right]. \tag{5}$$

The case $\beta = 0$ correspond to a fixed cost for each discretization point, and the case $\beta = 1$ corresponds to proportional costs.

In our framework, a discretization strategy will be said to be asymptotically optimal for a given cost functional if no strategy has (asymptotically, for large costs) a lower discretization error and a smaller cost.

Motivated by the form of the explicit asymptotically optimal strategy found by Fukasawa [1] and the readjustment rules used by market practitioners, we consider discretizations based on the hitting times of the process X. Such a discretization rule is defined by a pair of positive $\mathbb{F}$-adapted càdlàg processes $(\overline{a}_t)_{t\ge 0}$ and $(\underline{a}_t)_{t\ge 0}$. The discretization dates are then given by

$$T_{i+1}^\varepsilon = \inf\{t > T_i^\varepsilon : X_t \notin (X_{T_i^\varepsilon} - \varepsilon \underline{a}_{T_i^\varepsilon}, X_{T_i^\varepsilon} + \varepsilon \overline{a}_{T_i^\varepsilon})\}.$$

3. We characterize explicitly the asymptotic behavior of the errors and costs associated to our random discretization rules, by showing that, under suitable assumptions,

$$\lim_{\varepsilon\to 0} \varepsilon^{-2}\mathcal{E}(\varepsilon) = E\left[\int_0^T A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right] \tag{6}$$

$$\lim_{\varepsilon\to 0} \varepsilon^{\alpha-\beta}\mathcal{C}^\beta(\varepsilon) = E\left[\int_0^T \lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} dt\right], \tag{7}$$

where, for $\underline{a}, \overline{a} \in (0,\infty)$,

$$f(\underline{a}, \overline{a}) = E\left[\int_0^{\tau^*} (X_t^*)^2 dt\right], \quad g(\underline{a}, \overline{a}) = E[\tau^*] \quad \text{and} \quad u^\beta(\underline{a}, \overline{a}) = E[|X^*_{\tau^*}|^\beta] < \infty.$$

with $\tau^* = \inf\{t \ge 0 : X_t^* \notin (-\underline{a}, \overline{a})\}$, where X^* is a strictly α-stable process with Lévy density

$$\nu^*(x) = \frac{c_+ 1_{x>0} + c_- 1_{x<0}}{|x|^{1+\alpha}}.$$

The above result allows to prove that we may look for optimal barriers $\underline{a}$ and $\overline{a}$ as minimizers of

$$\min\left\{A_t \frac{f(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)} + c\lambda_t \frac{u^\beta(\underline{a}_t, \overline{a}_t)}{g(\underline{a}_t, \overline{a}_t)}\right\}. \tag{8}$$

The parameter c may be chosen by the trader depending on the maximum acceptable cost: the bigger c, the smaller will be the cost of the strategy and, consequently the bigger its error. The functions f, g and u appearing above must in general be computed numerically, however, in the case when the limiting process X^* is a symmetric stable process, which corresponds for example to the CGMY model very popular in practice, the results are completely explicit, as will be shown in the next paragraph.

4. Assume that Y is an exponential of a Lévy process: $Y_t = e^{Z_t}$ where Z is a Lévy process without diffusion part, and whose Lévy measure has a density $\nu(x)$ satisfying $\nu(x) \sim \frac{c}{|x|^{1+\alpha}}$, $x \to 0$. Then $A_t = \Sigma Y_t^2$ with $\Sigma = \int (e^z - 1)^2 \nu(z) dz$. The quadratic hedging strategy in this case has been given by several authors and is known to have a Markov structure: $X_t = \phi(t, Y_t)$ for a deterministic function ϕ. In this case we may compute

$$\lambda_t = \left(Y_t \frac{\partial \phi(t, Y_t)}{\partial Y} \right)^{\alpha}$$

and therefore

$$a_t = c \left(\frac{\partial \phi(t, Y_t)}{\partial Y} \right)^{\frac{\alpha}{2+\alpha-\beta}} Y_t^{\frac{\alpha-2}{\alpha-\beta+2}}.$$

When $\beta = 0$ and $\alpha \to 2$, we find that the optimal size of the rebalancing interval is proportional to the square root of $\frac{\partial \phi(t,Y_t)}{\partial Y}$ (the gamma), which is consistent with the results of Fukasawa [1], quoted above. In the general case, we obtain an explicit representation for the optimal discretization dates, which includes two "tuning" parameters: the index β which determines the effect of transaction costs (fixed, proportional, etc.) and the Blumenthal-Getoor index α measuring the activity of small jumps.

References

[1] M. Fukasawa, *Asymptotically efficient discrete hedging*, in Stochastic Analysis with Financial Applications, Progress in Probability, vol. 65, Springer, 2011.